A note on linear parallel displacements in Finsler geometry

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Abstract

The author states the necessary and sufficient condition that linear parallel displacements are symmetric for any curve and path, respectively. Further the author gives the determinant of linear transformations derived from linear parallel displacements and states a condition for the linear transformation to be an orthogonal one.

Keywords: linear parallel displacement, path, Finsler spaces.

Introduction

The author stated the notion of parallel displacements of vector fields along a curve on the 43-rd Symposium on Finsler geometry at Utsunomiya, 2008. The definition is linear with respect to a vector field. So the author redefines it as "linear" parallel displacement followed in [2] (In this book, however, linear parallel displacements are defined by an non-linear connection N). On the other hand, the traditional ways of the definition is not linear with respect to a vector field([1]). Further in the early stages of Finsler geometry, many authors gave its definitions([3]) and in the book [4] they were collected. In all cases, however, the studies that the author states in this paper were not investigated.

In the first half of section 1, the same contents to [7] and [8] with a little difference are stated and in the second half the necessary and sufficient condition that linear parallel displacements are symmetric for any path is stated. In section 2, the determinant of Φ derived from linear parallel displacements is given and cases that Φ is Identity transformation and an orthogonal one are stated.

The terminology and notations are referred to the books [5] and [6]. The author is given very useful suggestions by Prof.T.Aikou and Prof.M.Hashiguchi frequently, and greatly appreciates their kindness.

1 The definition of a linear parallel displacement along a curve

The contents of this section is involved [7]. Firstly, we put terminology and notations used in this paper. Let M be an n-dimensional differentiable manifold and $F\Gamma = (N^i_j(x,y), F^i_{jr}(x,y), C^i_{jr}(x,y))$ Finsler connection(or the coefficients of a Finsler connection $F\Gamma$) and all of objects appeared in this paper (curve, path, vector field, etc) are differentiable. In additions, indexes i,j,k,r,m,l,$\cdots$ run on 1 to n.

Now, for a vector field along a curve c, we give a following definition of linear parallel displacements along the curve.

Definition 1.1 *For a curve $c = (c^i(t))$ $(a \leq t \leq b)$ and a vector field $v = (v^i(t))$ along c, if the equation*

$$\frac{dv^i}{dt} + v^j F^i_{jr}(c, \dot{c})\dot{c}^r = 0 \quad (\dot{c}^r = \frac{dc^r}{dt}) \tag{1.1}$$

is satisfied, then v is called a parallel vector field along c, and we call the linear map $\Pi_c : v(a) \longrightarrow v(b)$ a linear parallel displacement along c.

Remark 1.1 *We can see that the differential equation* (1.1) *is linear with respect to a vector field v, so a linear parallel displacement Π_c is regular, namely, one to one and on to, because of the uniqueness of the solution of the differential equation* (1.1).

Let set the state as a curve $c = (c^i(t))$ $(a \leq t \leq b)$ passes through two points $p = c(a)$, $q = c(b)$ on M, and we assume that a vector field $v = (v^i(t))$ is parallel along c and $A = (A^i) = v(a)$, $B = (B^i) = v(b)$. Then, in general, we can have another curve c^{-1} and vector field v^{-1} as follows

$$c^{-1}(\tau) = (c^{-1i}(\tau)), \text{ where } c^{-1i}(\tau) = c^i(-\tau + a + b), \tag{1.2}$$

$$v^{-1}(\tau) = (v^{-1i}(\tau)), \text{ where } v^{-1i}(\tau) = v^i(-\tau + a + b) \tag{1.3}$$

and $t = -\tau + a + b$, $a \leq \tau \leq b$.
Then $c^{-1}(a) = c(b) = q$, $c^{-1}(b) = c(a) = p$ and $v^{-1}(a) = v(b) = B$, $v^{-1}(b) = v(a) = A$.

In general, the vector field v^{-1} is not parallel along the curve c^{-1}. Because the satisfying equation is, from $\dot{c}^{-1r} = -\dot{c}^r$, $\frac{dv^{-1i}}{d\tau} = -\frac{dv^i}{dt}$ and (1.1),

$$\frac{dv^{-1i}}{d\tau} + v^{-1j} F^i_{jr}(c^{-1}, -\dot{c}^{-1})\dot{c}^{-1r} = 0 \quad (\dot{c}^{-1r} = \frac{dc^{-1r}}{d\tau}). \tag{1.4}$$

So we consider a parallel vector field $u = (u^i(\tau))$ along c^{-1} with the initial value $u(a) = C$ and the end value $u(b) = A$. By the definition, $u(\tau)$ satisfies the equation

$$\frac{du^i}{d\tau} + u^j F^i_{jr}(c^{-1}, \dot{c}^{-1})\dot{c}^{-1r} = 0. \tag{1.5}$$

In addition, the vector field $u^{-1}(t)$ along the curve c satisfies

$$\frac{du^{-1i}}{dt} + u^{-1j} F^i_{jr}(c, -\dot{c})\dot{c}^r = 0 \quad (\dot{c}^r = \frac{dc^r}{dt}). \tag{1.6}$$

Then we investigate a transformation

Definition 1.2 $\Phi_{c,\dot{c}}(t) \; : \; v(t) \longrightarrow u^{-1}(t)$ *on* $T_{c(t)}M \quad \forall t \in [a, b]$

at every point $c(t)$ on the curve c, where the initial value $\Phi_{c,\dot{c}}(a) = Identity$(In the Riemannian case, $\Phi_{c,\dot{c}}(t)$ is the identity transformation, equivalently.).

Since the linearity of the equations (1.5) with respect to u, $\Phi_{c,\dot{c}}(t)$ is a linear transformation. Hence we have a $(1,1)$-tensor fields $\Phi^i_j(t)$ with the parameter t and $\Phi^i_j(t)$ satisfies the following equation

$$u^{-1i}(t) = \Phi^i_j(t)v^j(t). \tag{1.7}$$

Since $u^{-1i}(t)$ satisfies the equation (1.6) and (1.7),

$$\frac{d\Phi^i_j}{dt}v^j + \Phi^i_j\frac{dv^j}{dt} + \Phi^m_j v^j F^i_{mr}(c, -\dot{c})\dot{c}^r = 0 \tag{1.8}$$

is satisfied.

On the other hand v^j satisfies the equation (1.1). From (1.1) and (1.8), we have

$$\left(\frac{d\Phi^i_j}{dt} - \Phi^i_m F^m_{jr}(c, \dot{c})\dot{c}^r + \Phi^m_j F^i_{mr}(c, -\dot{c})\dot{c}^r\right) v^j = 0. \tag{1.9}$$

Since the arbitrariness of the vector field v, we have

$$\frac{d\Phi^i_j}{dt} - \Phi^i_m F^m_{jr}(c, \dot{c})\dot{c}^r + \Phi^m_j F^i_{mr}(c, -\dot{c})\dot{c}^r = 0. \tag{1.10}$$

So we have

Proposition 1.1 *For the transformation $\Phi_{c,\dot{c}}(t)$ defined by Definition 1.2, we put the components $\Phi^i_j(t)$, then $\Phi^i_j(t)$ satisfies (1.10).*

Now, if, on the curve c, $v = u^{-1}$ is satisfied, then $\Phi^i_j(t) = \delta^i_j$ are satisfied. Therefore $\frac{d\Phi^i_j}{dt} = 0$ are satisfied. Then we have, from (1.10),

$$F^i_{jr}(c, \dot{c})\dot{c}^r + F^i_{jr}(c, -\dot{c})(-\dot{c}^r) = 0. \tag{1.11}$$

Further we assume that the Finsler torsion tensor field T vanishes, namely, $T^i_{jr}(x, y) = F^i_{jr}(x, y) - F^i_{rj}(x, y) = 0$. Then we have, from (1.11)

$$F^i_{0j}(c, \dot{c}) + F^i_{0j}(c, -\dot{c}) = 0. \tag{1.12}$$

Inversely, we assume that $F^i_{0j}(c, \dot{c}) + F^i_{0j}(c, -\dot{c}) = 0$ is satisfied. Then we can prove that the inverse vector u^{-1} is parallel along the curve c as follows

$$\begin{aligned}
\frac{du^{-1i}}{dt} + u^{-1j}F^i_{jr}(c, -\dot{c})\dot{c}^r &= \frac{du^{-1i}}{dt} + u^{-1j}F^i_{rj}(c, -\dot{c})\dot{c}^r \\
&= \frac{du^{-1i}}{dt} - u^{-1j}F^i_{rj}(c, -\dot{c})(-\dot{c}^r) \\
&= \frac{du^{-1i}}{dt} + u^{-1j}F^i_{0j}(c, \dot{c}) \\
&= \frac{du^{-1i}}{dt} + u^{-1j}F^i_{rj}(c, \dot{c})\dot{c}^r \\
&= \frac{du^{-1i}}{dt} + u^{-1j}F^i_{jr}(c, \dot{c})\dot{c}^r.
\end{aligned} \tag{1.13}$$

The right hand side is equivalent to the left hand side of (1.1) of the parallel vector field v along c. From $v(a) = u^{-1}(a) = A$, we have

$$u^{-1} = v \text{ on } c. \tag{1.14}$$

Therefore

$$u = v^{-1} \text{ on } c^{-1} \tag{1.15}$$

is satisfied. From (1.15), we can see that v^{-1} is parallel along c^{-1} because u is parallel along c^{-1}. So we put a definition as follows

Definition 1.3 *Let v be a parallel vector field along a curve c. If v^{-1} is also parallel along the curve c^{-1}, then the linear parallel displacement Π_c is called symmetric.*

Then we have

Proposition 1.2 *Let c be a differentiable curve on M and the Finsler torsion tensor field T vanishes on c. The linear parallel displacement Π_c is symmetric if and only if the equation (1.12) is satisfied.*

And

Theorem 1.1 *Let M be an n-dimensional differentiable manifold with a Finsler connection $F\Gamma = (N^i_j(x,y),\ F^i_{jr}(x,y),\ C^i_{jr}(x,y))$ satisfying $T^i_{jr}(x,y) = 0$. For any differentiable curve c on M, the linear parallel displacement Π_c is symmetric if and only if $F^i_{0j}(x,y) + F^i_{0j}(x,-y) = 0$ is satisfied on M.*

Remark 1.2 *The quantity $F^i_{0j}(x,y) + F^i_{0j}(x,-y)$ is a Finsler tensor field.*

Hereafter, we put $H^i_j(x,y) \equiv F^i_{0j}(x,y) + F^i_{0j}(x,-y)$.

Next, we investigate the case of linear parallel displacements on paths. A path $c(t)$ is a curve satisfying the following differential equation

$$\frac{d\dot{c}^i}{dt} + N^i_r(c,\dot{c})\dot{c}^r = 0. \tag{1.16}$$

In here, we assume $N^i_0(x,y) = N^i_0(x,-y)$. This means that if a curve c is a path, then the inverse curve c^{-1} is also one.

In additions, we prepare notations for calculations.
If a quantity A have (x,y), then we put ${}^{+}A$. On the other hand, if a quantity A have $(x,-y)$, then we put ${}^{-}A$. And the contractions of y and $-y$ are as follows, for examples,

$${}^{+}A^i_0 = A^i_j(x,y)y^j, \quad {}^{-}A^i_0 = A^i_j(x,-y)(-y^j) = -A^i_j(x,-y)y^j.$$

We calculate $\frac{d\ {}^{+}H^i_j}{dt} = \frac{d}{dt}(\ {}^{+}F^i_{0j} + \ {}^{-}F^i_{0j})$ on a path $c(t)$, where ${}^{+}H^i_j = \ {}^{-}H^i_j$ and ${}^{+}N^i_0 = \ {}^{-}N^i_0$ are satisfied.

$$
\begin{aligned}
\frac{d}{dt}(\ {}^{+}F^i_{0j}) &= \frac{d}{dt}(\ {}^{+}F^i_{rj}(c,\dot{c})\dot{c}^r) = \frac{\partial\ {}^{+}F^i_{rj}}{\partial x^k}\frac{dc^k}{dt}\dot{c}^r + \frac{\partial\ {}^{+}F^i_{rj}}{\partial y^k}\frac{d\dot{c}^k}{dt}\dot{c}^r + \ {}^{+}F^i_{rj}\frac{d\dot{c}^r}{dt} \\
&= \left(\frac{\delta\ {}^{+}F^i_{rj}}{\delta x^k} + \ {}^{+}N^l_k\frac{\partial\ {}^{+}F^i_{rj}}{\partial y^l}\right)\dot{c}^k\dot{c}^r + \left(\frac{\partial\ {}^{+}F^i_{rj}}{\partial y^k}\dot{c}^r + \ {}^{+}F^i_{kj}\right)\frac{d\dot{c}^k}{dt} \\
&= \frac{\delta\ {}^{+}F^i_{rj}}{\delta x^k}\dot{c}^k\dot{c}^r + \ {}^{+}N^l_0\frac{\partial\ {}^{+}F^i_{rj}}{\partial y^l}\dot{c}^r + \left(\frac{\partial\ {}^{+}F^i_{rj}}{\partial y^k}\dot{c}^r + \ {}^{+}F^i_{kj}\right)(-\ {}^{+}N^k_0) \qquad (1.17) \\
&= \frac{\delta\ {}^{+}F^i_{rj}}{\delta x^k}\dot{c}^k\dot{c}^r + \ {}^{+}N^l_0\frac{\partial\ {}^{+}F^i_{rj}}{\partial y^l}\dot{c}^r - \ {}^{+}N^k_0\frac{\partial\ {}^{+}F^i_{rj}}{\partial y^k}\dot{c}^r - \ {}^{+}F^i_{kj}\ {}^{+}N^k_0 \\
&= \frac{\delta\ {}^{+}F^i_{rj}}{\delta x^k}\dot{c}^k\dot{c}^r - \ {}^{+}F^i_{kj}\ {}^{+}N^k_0.
\end{aligned}
$$

By the same manner, we have

$$
\frac{d}{dt}(\ {}^{-}F^i_{0j}) = -\frac{\delta\ {}^{-}F^i_{rj}}{\delta x^k}\dot{c}^k\dot{c}^r + \ {}^{-}F^i_{kj}\ {}^{-}N^k_0. \tag{1.18}
$$

Therefore we have

$$
\frac{d\ {}^{+}H^i_j}{dt}(c,\dot{c}) = \frac{\delta\ {}^{+}F^i_{rj}}{\delta x^k}\dot{c}^k\dot{c}^r - \ {}^{+}F^i_{kj}\ {}^{+}N^k_0 - \frac{\delta\ {}^{-}F^i_{rj}}{\delta x^k}\dot{c}^k\dot{c}^r + \ {}^{-}F^i_{kj}\ {}^{-}N^k_0. \tag{1.19}
$$

On the other hand, in general, the following equation

(1.20)

$$
\begin{aligned}
{}^{+}H^i_{j|0}(x,y) &= \left(\frac{\delta\ {}^{+}H^i_j}{\delta x^k} + \ {}^{+}H^l_j\ {}^{+}F^i_{lk} - \ {}^{+}H^i_l\ {}^{+}F^l_{jk}\right)y^k \\
&= \left(\frac{\delta}{\delta x^k}(\ {}^{+}F^i_{rj}y^r - \ {}^{-}F^i_{rj}y^r) + (\ {}^{+}F^l_{rj}y^r - \ {}^{-}F^l_{rj}y^r)\ {}^{+}F^i_{lk} - (\ {}^{+}F^i_{rl}y^r - \ {}^{-}F^i_{rl}y^r)\ {}^{+}F^l_{jk}\right)y^k \\
&= \frac{\delta\ {}^{+}F^i_{rj}}{\delta x^k}y^r y^k - \ {}^{+}F^i_{rj}\ {}^{+}N^r_0 - \frac{\delta\ {}^{-}F^i_{rj}}{\delta x^k}y^r y^k + \ {}^{-}F^i_{rj}\ {}^{-}N^r_0 + \ {}^{+}F^i_{0l}\ {}^{-}F^l_{0j} - \ {}^{-}F^i_{0l}\ {}^{+}F^l_{0j}
\end{aligned}
$$

is satisfied. Here we put ${}^{+}K^i_j(x,y) = \ {}^{+}F^i_{0l}\ {}^{-}F^l_{0j} - \ {}^{-}F^i_{0l}\ {}^{+}F^l_{0j}$. Then we have

$$
{}^{+}H^i_{j|0}(x,y) - \ {}^{+}K^i_j(x,y) = \frac{\delta\ {}^{+}F^i_{rj}}{\delta x^k}y^r y^k - \ {}^{+}F^i_{rj}\ {}^{+}N^r_0 - \frac{\delta\ {}^{-}F^i_{rj}}{\delta x^k}y^r y^k + \ {}^{-}F^i_{rj}\ {}^{-}N^r_0. \tag{1.21}
$$

If a path $c(t)$ is given, then, from (1.19) and (1.21),

$$
{}^{+}H^i_{j|0}(c,\dot{c}) - \ {}^{+}K^i_j(c,\dot{c}) = \frac{d\ {}^{+}H^i_j}{dt}(c,\dot{c}) \tag{1.22}
$$

is satisfied on the path $c(t)$.

Here, we assume ${}^{+}H^i_j(c,\dot{c}) = {}^{+}F^i_{0j}(c,\dot{c}) + {}^{-}F^i_{0j}(c,-\dot{c}) = 0$ on any path $c(t)$, namely, the linear parallel displacement Π_c is symmetric on $c(t)$. Obviously, $\frac{d\ {}^{+}H^i_j}{dt}(c,\dot{c}) = 0$ is satisfied. From (1.22),

$$(1.23) \qquad {}^{+}H^i_{j|0}(c,\dot{c}) - {}^{+}K^i_j(c,\dot{c}) = 0$$

holds good. In addition, from ${}^{-}F^i_{0j}(c,-\dot{c}) = -{}^{+}F^i_{0j}(c,\dot{c})$,

$$(1.24) \qquad {}^{+}K^i_j(c,\dot{c}) = 0$$

is satisfied. From (1.23) and (1.24), we have

$$(1.25) \qquad {}^{+}H^i_{j|0}(c,\dot{c}) = 0.$$

Now, we have a solution of the path equation (1.16) at arbitrary points $(c = x, \dot{c} = y)$. So (1.24) and (1.25) are satisfied at any point (x,y), namely, we have

$$(1.26) \qquad {}^{+}H^i_{j|0}(x,y) = 0 \text{ and } {}^{+}K^i_j(x,y) = 0.$$

Inversely, if we assume (1.26), then on any path $c(t)$, ${}^{+}H^i_{j|0}(c,\dot{c}) = 0$ and ${}^{+}K^i_j(c,\dot{c}) = 0$ are satisfied. So, from (1.22), we have $\frac{d\ {}^{+}H^i_j}{dt}(c,\dot{c}) = 0$. Therefore ${}^{+}H^i_j(c,\dot{c}) = {}^{+}F^i_{0j}(c,\dot{c}) + {}^{-}F^i_{0j}(c,-\dot{c}) = \lambda^i_j$ (constant) holds good on the path $c(t)$. We notice the quantity ${}^{+}H^i_j$ is a Finsler tensor field. By any coordinate transformation $\bar{x} \longrightarrow x$, $\bar{\lambda}^i_j = \lambda^r_k \frac{\partial \bar{x}^i}{\partial x^r} \frac{\partial x^k}{\partial \bar{x}^j}$ is satisfied. However, $\bar{\lambda}^i_j$ have to be constant with respect to the parameter t. Therefore $\lambda^i_j = 0$ ought to be satisfied. So we have

$$(1.27) \qquad {}^{+}H^i_j(c,\dot{c}) = {}^{+}F^i_{0j}(c,\dot{c}) + {}^{-}F^i_{0j}(c,-\dot{c}) = 0.$$

Namely, on the path $c(t)$, the linear parallel displacement Π_c is symmetric.

Further, we have, by the same calculations,

$$(1.28) \qquad {}^{+}H^i_j(x,y) = {}^{-}H^i_j(x,-y),$$

$$(1.29) \qquad \frac{d\ {}^{-}H^i_j}{dt}(c,-\dot{c}) = \frac{d\ {}^{+}H^i_j}{dt}(c,\dot{c}) \text{ (on any path } c(t)),$$

$$(1.30) \qquad {}^{-}K^i_j(x,-y) = -{}^{+}K^i_j(x,y),$$

$$(1.31) \qquad {}^{-}H^i_{j|0}(x,-y) - {}^{-}K^i_j(x,-y) = -{}^{+}H^i_{j|0}(x,y) + {}^{+}K^i_j(x,y),$$

$$(1.32) \qquad {}^{-}H^i_{j|0}(c,-\dot{c}) - {}^{-}K^i_j(c,-\dot{c}) = -\frac{d\ {}^{-}H^i_j}{dt}(c,-\dot{c}) \text{ (on any path } c(t)).$$

So " ${}^{-}H^i_{j|0}(x,-y) = 0$ and ${}^{-}K^i_j(x,-y) = 0$" is equivalent to ${}^{-}H^i_j(c,-\dot{c})(= {}^{+}H^i_j(c,\dot{c})) = 0$ on any path $c(t)$. Consequently, we have

Theorem 1.2 *Let M be an n-dimensional differentiable manifold with a Finsler connection $F\Gamma = (N^i_j(x,y),\ F^i_{jr}(x,y),\ C^i_{jr}(x,y))$ satisfying $T^i_{jr}(x,y) = 0$ and $N^i_0(x,y) = N^i_0(x,-y)$. For any path $c(t)$ on M, the linear parallel displacement Π_c is symmetric on $c(t)$ if and only if $H^i_{j|0}(x,y) = 0$ and $K^i_j(x,y) = 0$ are satisfied.*

Remark 1.3 *The quantity $H^i_{j|0}(x,y)$ is a Finsler tensor field, however, $K^i_j(x,y)$ is not one. But the vanishing property $K^i_j(x,y)=0$ is independent of the choice of coordinate neighborhoods as follows:*
For coordinate neighborhoods $\{U,(x,y)\}, \{\bar{U},(\bar{x},\bar{y})\}$, where $U\cap\bar{U}\neq\phi$,

$$\bar{K}^i_j(\bar{x},\bar{y}) = K^l_q(x,y)\frac{\partial\bar{x}^i}{\partial x^l}\frac{\partial x^q}{\partial\bar{x}^j} - H^l_m(x,y)\left(\frac{\partial\bar{x}^i}{\partial x^l}\frac{\partial^2 x^m}{\partial\bar{x}^j\partial\bar{x}^k} + \frac{\partial x^m}{\partial\bar{x}^j}\frac{\partial x^p}{\partial\bar{x}^k}\frac{\partial^2\bar{x}^i}{\partial x^l\partial x^p}\right)\bar{y}^k \tag{1.33}$$

holds good. So, on U, if $H^i_{j|0}(x,y)=0$ and $K^i_j(x,y)=0$ are satisfied, then $H^i_j(c,\dot{c})=0$ is satisfied on any path $c(t)$. Therefore, on $\bar{U}$, $\bar{K}^i_j(\bar{x},\bar{y})=0$ is satisfied.

2 The determinant of $\Phi_{c,\dot{c}}$

For a curve $c(t)$ $(a\leq t\leq b)$, the linear parallel displacement Π_c is regular as that is stated in the previous section, namely, the determinant $|\Pi_c|\neq 0$ is satisfied. By the way, the linear transformation $\Phi_{c,\dot{c}}$ is regarded as the inverse of a composite of Π_c and $\Pi_{c^{-1}}$ as follows

$$\Phi_{c,\dot{c}} = (\Pi_c\circ\Pi_{c^{-1}})^{-1}. \tag{2.1}$$

Therefore $\Phi_{c,\dot{c}}$ is regular. We rewrite the equation (1.10) under $T^i_{rj}=0$

$$\frac{d\Phi^i_j}{dt} = \Phi^i_m\,{}^{+}F^m_{0j} + \Phi^m_j\,{}^{-}F^i_{0m}, \tag{2.2}$$

where $\Phi^i_j=\Phi^i_j(t)$, ${}^{+}F^m_{0j}=F^m_{rj}(c(t),\dot{c}(t))\dot{c}^r(t)$, ${}^{-}F^i_{0m}=F^i_{rm}(c(t),-\dot{c}(t))(-\dot{c}^r(t))$.

We calculate the determinant $\det(\Phi^i_j)$ of the matrix (Φ^i_r). For the calculation, we have the following preliminaries.

$$f(t)\equiv\det(\Phi^i_j)=|\Phi^i_j| = \begin{vmatrix}\Phi^1_1 & \Phi^1_2 & \cdots & \Phi^1_n\\ \vdots & \vdots & \vdots & \vdots\\ \Phi^n_1 & \Phi^n_2 & \cdots & \Phi^n_n\end{vmatrix},\quad \Phi^{i'}_j\equiv\frac{d\Phi^i_j}{dt}, \tag{2.3}$$

$$X_k\equiv\begin{pmatrix}\Phi^1_k\\ \vdots\\ \Phi^n_k\end{pmatrix},\quad X'_k\equiv\begin{pmatrix}\Phi^{1'}_k\\ \vdots\\ \Phi^{n'}_k\end{pmatrix}\quad (k=1,\cdots,n). \tag{2.4}$$

Then we can rewrite the various quantities as follows

$$f(t) = |X_1\ X_2\cdots X_n|, \tag{2.5}$$

$$\Phi^{i'}_j = \Phi^i_m\,{}^{+}F^m_{0j} + \Phi^m_j\,{}^{-}F^i_{0m}, \tag{2.6}$$

$$X'_k = (\Phi^{i'}_k) = (\Phi^i_m\,{}^{+}F^m_{0k}) + (\Phi^m_k\,{}^{-}F^i_{0m}). \tag{2.7}$$

In general, the differential of the determinant $f(t)$ with respect to t is written by

$$f'(t) = |X'_1\ X_2\cdots X_n| + |X_1\ X'_2\ X_3\cdots X_n| + \cdots + |X_1\ \cdots X_{n-1}\ X'_n|. \tag{2.8}$$

We denote the k-th term of (2.8) by df_k and calculate it.

$$
\begin{aligned}
df_k &= |X_1 \ \cdots \ X_k{}' \cdots X_n| \\
&= |X_1 \ \cdots \ (\Phi^{i}_{k}{}') \cdots X_n| = |X_1 \ \cdots \ (\Phi^i_m \ {}^{+}F^m_{0k}) + (\Phi^m_k \ {}^{-}F^i_{0m}) \cdots X_n| \\
&= |X_1 \ \cdots \ (\Phi^i_m \ {}^{+}F^m_{0k}) \cdots X_n| + |X_1 \ \cdots \ (\Phi^m_k \ {}^{-}F^i_{0m}) \cdots X_n|.
\end{aligned} \tag{2.9}
$$

Further, from

$$
\begin{aligned}
(\Phi^i_m \ {}^{+}F^m_{0k}) &= \left(\Phi^i_1 \ {}^{+}F^1_{0k} + \cdots + \Phi^i_k \ {}^{+}F^k_{0k} + \cdots + \Phi^i_n \ {}^{+}F^n_{0k}\right) \ (k : \text{fixed}) \\
&= \begin{pmatrix} \Phi^1_1 \ {}^{+}F^1_{0k} + \cdots + \Phi^1_k \ {}^{+}F^k_{0k} + \cdots + \Phi^1_n \ {}^{+}F^n_{0k} \\ \Phi^2_1 \ {}^{+}F^1_{0k} + \cdots + \Phi^2_k \ {}^{+}F^k_{0k} + \cdots + \Phi^2_n \ {}^{+}F^n_{0k} \\ \vdots \\ \Phi^n_1 \ {}^{+}F^1_{0k} + \cdots + \Phi^n_k \ {}^{+}F^k_{0k} + \cdots + \Phi^n_n \ {}^{+}F^n_{0k} \end{pmatrix} \\
&= \begin{pmatrix} \Phi^1_1 \ {}^{+}F^1_{0k} \\ \Phi^2_1 \ {}^{+}F^1_{0k} \\ \vdots \\ \Phi^n_1 \ {}^{+}F^1_{0k} \end{pmatrix} + \cdots + \begin{pmatrix} \Phi^1_k \ {}^{+}F^k_{0k} \\ \Phi^2_k \ {}^{+}F^k_{0k} \\ \vdots \\ \Phi^n_k \ {}^{+}F^k_{0k} \end{pmatrix} + \cdots + \begin{pmatrix} \Phi^1_n \ {}^{+}F^n_{0k} \\ \Phi^2_n \ {}^{+}F^n_{0k} \\ \vdots \\ \Phi^n_n \ {}^{+}F^n_{0k} \end{pmatrix} \\
&= {}^{+}F^1_{0k} \left(\Phi^i_1\right) + \cdots + {}^{+}F^k_{0k} \left(\Phi^i_k\right) + \cdots + {}^{+}F^n_{0k} \left(\Phi^i_n\right) \\
&= {}^{+}F^1_{0k} X_1 + \cdots + {}^{+}F^k_{0k} X_k + \cdots + {}^{+}F^n_{0k} X_n,
\end{aligned} \tag{2.10}
$$

we can see the first term of the right hand side of (2.9) by (2.10) as follows

(2.11)

$$
\begin{aligned}
&|X_1 \ \cdots \ (\Phi^i_m \ {}^{+}F^m_{0k}) \cdots X_n| \\
&= |X_1 \ \cdots \ {}^{+}F^1_{0k} X_1 + \cdots + {}^{+}F^k_{0k} X_k + \cdots + {}^{+}F^n_{0k} X_n \cdots X_n| \\
&= |X_1 \ \cdots \ {}^{+}F^1_{0k} X_1 \cdots X_n| + \cdots + |X_1 \ \cdots \ {}^{+}F^k_{0k} X_k \cdots X_n| + \cdots + |X_1 \ \cdots \ {}^{+}F^n_{0k} X_n \cdots X_n| \\
&= {}^{+}F^1_{0k} |X_1 \ \cdots \ X_1 \cdots X_n| + \cdots + {}^{+}F^k_{0k} |X_1 \ \cdots \ X_k \cdots X_n| + \cdots + {}^{+}F^n_{0k} |X_1 \ \cdots \ X_n \cdots X_n| \\
&= {}^{+}F^k_{0k} |X_1 \ \cdots \ X_k \cdots X_n| = {}^{+}F^k_{0k} f(t) \ \ (k : \text{fixed}).
\end{aligned}
$$

And, from

$$
\begin{aligned}
(\Phi^m_k \ {}^{-}F^i_{0m}) &= \left(\Phi^1_k \ {}^{-}F^i_{01} + \cdots + \Phi^k_k \ {}^{-}F^i_{0k} + \cdots + \Phi^n_k \ {}^{-}F^i_{0n}\right) \ (k : \text{fixed}) \\
&= \begin{pmatrix} \Phi^1_k \ {}^{-}F^1_{01} + \cdots + \Phi^k_k \ {}^{-}F^1_{0k} + \cdots + \Phi^n_k \ {}^{-}F^1_{0n} \\ \Phi^1_k \ {}^{-}F^2_{01} + \cdots + \Phi^k_k \ {}^{-}F^2_{0k} + \cdots + \Phi^n_k \ {}^{-}F^2_{0n} \\ \vdots \\ \Phi^1_k \ {}^{-}F^n_{01} + \cdots + \Phi^k_k \ {}^{-}F^n_{0k} + \cdots + \Phi^n_k \ {}^{-}F^n_{0n} \end{pmatrix} \\
&= \begin{pmatrix} \Phi^1_k \ {}^{-}F^1_{01} \\ \Phi^1_k \ {}^{-}F^2_{01} \\ \vdots \\ \Phi^1_k \ {}^{-}F^n_{01} \end{pmatrix} + \cdots + \begin{pmatrix} \Phi^k_k \ {}^{-}F^1_{0k} \\ \Phi^k_k \ {}^{-}F^2_{0k} \\ \vdots \\ \Phi^k_k \ {}^{-}F^n_{0k} \end{pmatrix} + \cdots + \begin{pmatrix} \Phi^n_k \ {}^{-}F^1_{0n} \\ \Phi^n_k \ {}^{-}F^2_{0n} \\ \vdots \\ \Phi^n_k \ {}^{-}F^n_{0n} \end{pmatrix} \\
&= \Phi^1_k \left({}^{-}F^i_{01}\right) + \cdots + \Phi^k_k \left({}^{-}F^i_{0k}\right) + \cdots + \Phi^n_k \left({}^{-}F^i_{0n}\right),
\end{aligned} \tag{2.12}
$$

we can see the second term of the right hand side of (2.9) by (2.12) as follows

(2.13)

$$
\begin{aligned}
&|X_1 \ \cdots \ (\Phi^m_k \ {}^-F^i_{0m}) \cdots X_n| \\
&= |X_1 \ \cdots \ \Phi^1_k \left({}^-F^i_{01} \right) + \cdots + \Phi^k_k \left({}^-F^i_{0k} \right) + \cdots + \Phi^n_k \left({}^-F^i_{0n} \right) \cdots X_n| \\
&= |X_1 \ \cdots \ \Phi^1_k \left({}^-F^i_{01} \right) \cdots X_n| + \cdots + |X_1 \ \cdots \ \Phi^k_k \left({}^-F^i_{0k} \right) \cdots X_n| + \cdots + |X_1 \ \cdots \ \Phi^n_k \left({}^-F^i_{0n} \right) \cdots X_n| \\
&= \Phi^1_k |X_1 \ \cdots \ \left({}^-F^i_{01} \right) \cdots X_n| + \cdots + \Phi^k_k |X_1 \ \cdots \ \left({}^-F^i_{0k} \right) \cdots X_n| + \cdots + \Phi^n_k |X_1 \ \cdots \ \left({}^-F^i_{0n} \right) \cdots X_n| \\
&\ (k : \text{fixed}).
\end{aligned}
$$

Then we have the form of the k-th term df_k as follows

$$
df_k = {}^+F^k_{0k} f(t) + \sum_{r=1}^{n} \Phi^r_k |X_1 \ \cdots \ X_{k-1} \ \left({}^-F^i_{0r} \right) \ X_{k+1} \cdots X_n| \quad (k : \text{fixed}). \tag{2.14}
$$

Therefore the differential $f'(t) = \sum_{k=1}^{n} df_k$ of the determinant $f(t)$ is written by

$$
f'(t) = \sum_{k=1}^{n} {}^+F^k_{0k} f(t) + \sum_{k=1}^{n} \sum_{r=1}^{n} \Phi^r_k |X_1 \ \cdots \ X_{k-1} \ \left({}^-F^i_{0r} \right) \ X_{k+1} \cdots X_n|. \tag{2.15}
$$

Next, we investigate the second term of (2.15).

(2.16)

$$
\begin{aligned}
&\sum_{k=1}^{n} \sum_{r=1}^{n} \Phi^r_k |X_1 \ \cdots \ X_{k-1} \ \left({}^-F^i_{0r} \right) \ X_{k+1} \cdots X_n| \\
&= \Phi^1_1 | \left({}^-F^i_{01} \right) \ X_2 \ \cdots \ X_n| + \cdots + \Phi^k_1 | \left({}^-F^i_{0k} \right) \ X_2 \ \cdots \ X_n| + \cdots + \Phi^n_1 | \left({}^-F^i_{0n} \right) \ X_2 \ \cdots \ X_n| \\
&+ \Phi^1_2 |X_1 \ \left({}^-F^i_{01} \right) \ \cdots \ X_n| + \cdots + \Phi^k_2 |X_1 \ \left({}^-F^i_{0k} \right) \ \cdots \ X_n| + \cdots + \Phi^n_2 |X_1 \ \left({}^-F^i_{0n} \right) \ \cdots \ X_n| \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&+ \Phi^1_n |X_1 \ \cdots \ X_{n-1} \ \left({}^-F^i_{01} \right) | + \cdots + \Phi^k_n |X_1 \ \cdots \ X_{n-1} \ \left({}^-F^i_{0k} \right) | + \cdots + \Phi^n_n |X_1 \ \cdots \ X_{n-1} \ \left({}^-F^i_{0n} \right) |.
\end{aligned}
$$

To calculate the above equation we consider the following n-systems of linear equations.

$$
\begin{pmatrix}
\Phi^1_1 & \Phi^1_2 & \cdots & \Phi^1_n \\
\Phi^2_1 & \Phi^2_2 & \cdots & \Phi^2_n \\
\vdots & \vdots & \vdots & \vdots \\
\Phi^n_1 & \Phi^n_2 & \cdots & \Phi^n_n
\end{pmatrix}
\begin{pmatrix}
x^1 \\ x^2 \\ \vdots \\ x^n
\end{pmatrix}
=
\begin{pmatrix}
{}^-F^1_{0k} \\ {}^-F^2_{0k} \\ \vdots \\ {}^-F^n_{0k}
\end{pmatrix}
\quad (k = 1, 2, \cdots, n). \tag{2.17}
$$

This n-systems has a solution $(x^1, \cdots, x^n)$ because of the regularity of Φ. Then

$$
\Phi^i_1 x^1 + \Phi^i_2 x^2 + \cdots + \Phi^i_n x^n = {}^-F^i_{0k} \quad (i = 1, 2, \cdots, n) \tag{2.18}
$$

are satisfied. Therefore

$$
\Phi^i_1 x^1 f(t) + \Phi^i_2 x^2 f(t) + \cdots + \Phi^i_n x^n f(t) = {}^-F^i_{0k} f(t) \quad (i = 1, 2, \cdots, n) \tag{2.19}
$$

are satisfied.
Next, if we multiply the both side of (2.17) by the cofactor matrix of (Φ^i_j) then we have

$$\begin{pmatrix} f(t) & \cdots & \cdots & 0 \\ 0 & f(t) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & f(t) \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix} = \begin{pmatrix} |\,(\,{}^-F^i_{0k})\ X_2\ \cdots\ X_n| \\ |X_1\ (\,{}^-F^i_{0k})\ \cdots\ X_n| \\ \vdots \\ |X_1\ \cdots\ X_{n-1}\ (\,{}^-F^i_{0k})\,| \end{pmatrix}, \tag{2.20}$$

namely

$$x^j f(t) = |X_1\ \cdots X_{j-1}\ (\,{}^-F^i_{0k})\ X_{j+1}\cdots\ X_n|\ \ (j = 1, 2, \cdots, n) \tag{2.21}$$

are satisfied. Therefore we have, from (2.19) and (2.21),

(2.22)
$$\Phi^i_1|\,(\,{}^-F^j_{0k})\ X_2\ \cdots\ X_n| + \Phi^i_2|X_1\ (\,{}^-F^j_{0k})\ \cdots\ X_n| + \cdots + \Phi^i_n|X_1\ \cdots\ X_{n-1}\ (\,{}^-F^j_{0k})\,| = {}^-F^i_{0k}f(t)$$
$(i, k = 1, 2, \cdots, n)$

From (2.16) and (2.22), we have

$$\begin{aligned} &\sum_{k=1}^{n}\sum_{r=1}^{n} \Phi^r_k|X_1\ \cdots\ X_{k-1}\ (\,{}^-F^i_{0r})\ X_{k+1}\cdots X_n| \\ &= {}^-F^1_{01}f(t) + \cdots + {}^-F^k_{0k}f(t) + \cdots + {}^-F^n_{0n}f(t) \\ &= (\,{}^-F^1_{01} + \cdots + {}^-F^k_{0k} + \cdots + {}^-F^n_{0n})f(t) = \sum_{k=1}^{n} {}^-F^k_{0k}f(t) \end{aligned} \tag{2.23}$$

Therefore, we have, from (2.15) and (2.23),

$$f'(t) = \sum_{k=1}^{n}(\,{}^+F^k_{0k} + {}^-F^k_{0k})f(t) = trace(H^i_j(t))f(t), \tag{2.24}$$

where $H^i_j(t) = {}^+F^i_{0j}(t) + {}^-F^i_{0j}(t)$.

Let's solve the above differential equation (2.24). Then we have $f(t) = C\exp(\int_a^t traceH(\sigma)d\sigma)$ (C : constant). At the initial value $t = a$, $\Phi = Identity$. So $C = 1$ is true. Therefore we have

$$\det\Phi(t) = \exp(\int_a^t traceH(\sigma)d\sigma)\ \ (a \le t \le b). \tag{2.25}$$

From (2.25), we notice that $\det\Phi(t)$ is positive and Φ preserves the orientation of tangent spaces on $c(t)$. Further if the trace of H vanishes, then Φ is an orthogonal transformation. In addition, according to the previous section, if $H \equiv 0$ is satisfied, then obviously $\Phi \equiv Identity$ is true . Then we have

Theorem 2.1 *For a curve $c(t)$, the determinant of Φ defined by Definition 1.2 is given by (2.25). And Φ is a linear transformation which preserves the orientation of tangent spaces on $c(t)$ and satisfies (1) and (2) as follows:*
(1) if $H \equiv 0$, then $\Phi \equiv Identity$ on $c(t)$,
(2) if $traceH \equiv 0$, then all Φ are orthogonal transformations on $c(t)$

Remark 2.1 *Spaces which satisfy* (1) *are Riemannian spaces and Berwald ones. But we don't know examples satisfying* (2)*, yet.*

References

[1] T. Aikou and L. Kozma. Global aspects of Finsler geometry. In *Handbook of global analysis*, pages 1–39, 1211. Elsevier Sci. B. V., Amsterdam, 2008.

[2] S.-S. Chern and Z. Shen. *Riemann-Finsler geometry*, volume 6 of *Nankai Tracts in Mathematics*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.

[3] M. Hashiguchi. On parallel displacements in Finsler spaces. *J. Math. Soc. Japan*, 10:365–379, 1958.

[4] M. Matsumoto. *The theory of Finsler connections*. Publications of the Study Group of Geometry, Vol. 5. Department of Mathematics, Okayama University, Okayama, 1970.

[5] M. Matsumoto. *Foundations of Finsler geometry and special Finsler spaces*. Kaiseisha Press, Shigaken, 1986.

[6] M. Matsumoto. Finsler geometry in the 20th-century. In *Handbook of Finsler geometry. Vol. 1, 2*, pages 557–966. Kluwer Acad. Publ., Dordrecht, 2003.

[7] T. Nagano. Notes on the notion of the parallel displacement in Finsler geometry. *Tensor (N.S.)*, 70(3):302–310, 2008.

[8] T. Nagano. On the parallel displacement and parallel vector fields in finsler geometry. *Acta Math. Acad. Paedagog. Nyházi.*, 26(2):349–358, 2010.

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