The Stability of Machian Cosmological Models in the Scalar-Tensor Theory of Gravitation

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Abstract

The stability of Machian cosmological models to an infinitesimal perturbation of radius is discussed for the dust model in the Brans-Dicke theory, the perfect fluid model with extended (negative) pressure in the Brans-Dicke theory, and the barotropic evolution of matter with the variable coupling function $\omega(\phi)$ in the generalized scalar-tensor theory of gravitation. Permissible ranges of the coupling constant ω for the stability of individual cases are derived and the unknown parameter ω_0 of the proposed coupling function is shown to be 3. The unstability of Machian cosmological models means a violation of Machian constraints like $a(t)\phi(t) = const.$

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Recently we derived a series of cosmological models in the Brans-Dicke theory [1] of gravitation and in the generalized scalar-tensor theory of gravitation [2]-[4] in the framework of the Machian point of view, with pressure in the extended range (*negative pressure*) and an additional *varying cosmological constant* [5]-[10]. Furthermore, by introducing a scalar field φ as *dark-energy*, we discussed the scenario of the barotropic evolution of matter and our universe [5], [11]. We could understand the mechanism why the measured coupling parameter ω of the Brans-Dicke theory of gravitation is so large at present [12], due to the time-variation of the coupling function $\omega(\phi)$ in this Machian model. We also indicated that this cosmological model shows the slowly accelerating expansion of the universe suggested by the recent measurements [13] of distances to type Ia supernovae.

The basic feature of these cosmological models is that the Brans-Dicke scalar field has the asymptotic form $\phi = O(\rho/\omega)$. Therefore, when matter of the universe vanishes, the gravitational constant loses its meaning, that is, empty space does not have inertia. Similarly, if the coupling constant of the scalar field continuously diverges to infinity, then the coupling between the scalar field and matter of the universe is cut off and a particle in the universe loses inertia. It is believed that the field equations of the Brans-Dicke theory coincide with those of the Einstein theory with the same energy-momentum tensor when the coupling constant ω goes to infinity, but this is generally not true [5], [6]. It is only the case in which the scalar field has the asymptotic form $\phi = \langle\phi\rangle + O(1/\omega)$, where $\langle\phi\rangle$ is a constant, that the correspondence between General Relativity and the Brans-Dicke theory holds on. Our cosmological model without the cosmological constant in the Brans-Dicke theory is rather relevant, in the infinite (discrete) limit of the coupling constant, to the static Einstein universe including the *ad hoc* cosmological constant.

Another feature of our cosmological model is that the universe explicitly expands linearly. This fact is a little peculiar in gravitational phenomena. In the Einstein universe, the gravitational attractive force balances with the repulsive force by the introduced cosmological constant and thus the universe becomes static. The similar mechanism might occur to our models. In the balance between the gravitational force and the virtual cosmological constant produced by the Brans-Dicke scalar fields, the universe seems to expand uniformly because of the degree of freedom due to the time-varying cosmological constant.

However, it is well-known that the static Einstein universe is unstable to a perturbation of its radius. The Einstein universe collapses by an infinitesimal negative perturbation and inversely expands exponentially by an infinitesimal positive perturbation. Actually, the general solution of the Einstein field equations with the cosmological constant for the homogeneous and isotropic universe is the Lemaitre universe. In this short note, we discuss the stability of the Machian cosmological models for an infinitesimal perturbation of the expansion parameter in the Brans-Dicke theory of gravitation and the generalized scalar-tensor theory of gravitation.

First let us consider the simplest case and exemplify our method. The Brans-Dicke theory has a particular closed solution for the homogeneous and isotropic universe with dust ($p = 0$) [14]-[16],

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)]\,, \tag{1}$$

$$\phi(t) = -[8\pi/(3+2\omega)c^2]\rho(t)t^2\,, \tag{2}$$

$$a(t) = [-2/(2+\omega)]^{1/2}\,t\,, \tag{3}$$

$$2\pi^2 a^3(t)\rho(t) = M \tag{4}$$

for the coupling constant $\omega < -2$, where M is the total mass of the universe. This cosmological model satisfies the identity [14], [15]

$$a(t)\phi(t) = D = const\,, \tag{5}$$

from which we directly derive the exact Machian relation

$$G(t)M/c^2a(t) = \pi\,, \tag{6}$$

where the gravitational "constant" $G = (4+2\omega)/(3+2\omega)\phi$.

The nonvanishing components of the Brans-Dicke field equations for the Robertson-Walker metric are

$$2a\ddot{a} + \dot{a}^2 + 1 = -\frac{8\pi}{(3+2\omega)c^2}\frac{a^2\rho}{\phi} - \frac{1}{2}\omega a^2\left(\frac{\dot{\phi}}{\phi}\right)^2 + a\dot{a}\left(\frac{\dot{\phi}}{\phi}\right)\,, \tag{7}$$

$$\frac{3}{a^2}\left(\dot{a}^2+1\right) = \frac{16\pi(1+\omega)}{(3+2\omega)c^2}\frac{\rho}{\phi} + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\ddot{\phi}}{\phi}\,, \tag{8}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = \frac{8\pi}{(3+2\omega)c^2}\rho\,, \tag{9}$$

where a dot denotes the derivative with respect to t, for three unknown functions $a(t)$, $\phi(t)$, and $\rho(t)$. The equation (7) explicitly determines the acceleration rate of expansion $\ddot{a}$ of the universe but it is a little complicated. We usually adopt the conservation law of the energy-momentum, which is dynamically derived from the field equations themselves, as the independent equation instead of Eq.(7) and seek the solution Eqs.(2)-(4).

In this exact cosmological model, we investigate the behavior of an infinitesimal perturbation to the expansion parameter. Let us denote as

$$a(t) \longrightarrow a(t) + \delta a(t)\,. \tag{10}$$

We directly estimate the acceleration rate of the infinitesimal perturbation by means of Eq.(7) in the framework of the present cosmological model with the Machian constraint $a(t)\phi(t) = const.$ Generally speaking, this particular model might evolve as a more general cosmological solution, which is unknown, in the same way that the static Einstein universe generally evolves as the Lemaitre universe in General Relativity. We may restrict our discussions to Eq.(7) and neglect more general perturbations here. We obtain from the Machian constraint Eq.(5)

$$\frac{\dot{a}}{a} = -\frac{\dot{\phi}}{\phi}\,. \tag{11}$$

Using the relation derived from the Machian constraint and the conservation law of energy:

$$\frac{a^2\rho}{\phi} = \frac{M}{2\pi^2 D}\,, \tag{12}$$

we derive from Eq.(7)

$$2a\ddot{a} = -1 - \frac{4M}{(3+2\omega)\pi c^2 D} - \frac{1}{2}(4+\omega)\dot{a}^2\,. \tag{13}$$

Estimating some quantities by the non-perturbed solution, and neglecting higher orders of perturbation in δa, we find as the acceleration rate of the perturbed term

$$2a\delta\ddot{a} = -(4+\omega)\dot{a}\delta\dot{a}\,, \tag{14}$$

where $\delta\dot{a}$ and $\delta\ddot{a}$ denote $d(\delta a)/dt$ and $d^2(\delta a)/dt^2$ respectively. In the case that the coupling constant satisfies $4+\omega < 0$, if the derivative of the perturbed term $\delta\dot{a} > 0$, then the second-derivative of the perturbed term $\delta\ddot{a}$ becomes positive and inversely if $\delta\dot{a} < 0$, then $\delta\ddot{a} < 0$. This means that the induced infinitesimal perturbation increases or decreases exponentially like the static Einstein universe and we consider this cosmological model unstable. In case of $4+\omega > 0$, if $\delta\dot{a} > 0$, then $\delta\ddot{a} < 0$, or if $\delta\dot{a} < 0$, then $\delta\ddot{a} > 0$. In these conditions, the derivative of the perturbed term $\delta\dot{a}$ rapidly approaches to zero and also the second-derivative of the perturbed term $\delta\ddot{a}$ converges to zero. The induced infinitesimal perturbation remains constant and the ratio $\delta a(t)/a(t)$ diminishes as the universe expands. So we may consider this cosmological model stable. Thus for the stable Machian cosmological model for dust ($p = 0$), the coupling constant is restricted to the range

$$-4 < \omega < -2\,. \tag{15}$$

When the coupling constant ω diverges to minus infinity ($\omega \to -\infty$), the present Machian cosmological model in the Brans-Dicke theory converges to the static Einstein-like universe by means of redefinition of the initial condition for its radius (the isolated limit) [17]:

$$\lambda a^2 = 1\,, \quad \kappa\rho c^2 a^2 = 4\,, \tag{16}$$

and the field equations (7) and (8) reduce to

$$2a\ddot{a} + \dot{a}^2 - \lambda a^2 + 1 = 0\,, \tag{17}$$

$$\frac{3}{a^2}\left(\dot{a}^2 + 1\right) + \lambda = \kappa\rho c^2 \tag{18}$$

respectively, where the cosmological constant $\lambda \equiv -(\omega/2)(\dot{\phi}/\phi)^2$ and the Einstein's gravitational constant $\kappa \equiv 8\pi/c^4\phi$. It is confirmed that this reduced cosmological model is unstable for an infinitesimal perturbation. In general, the Machian cosmological model is unstable except the narrow range of the coupling constant ($-4 < \omega < -2$), which is favorable to the Machian point of view [1], in the above situation.

Next we extend our discussions to the perfect fluid case ($p \neq 0$). The energy conservation gives the equation of continuity

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + p/c^2\right) = 0\,, \tag{19}$$

and by considering the barotropic equation of state

$$p(t) = \gamma\rho(t)c^2\,, \tag{20}$$

we obtain directly

$$\rho(t)a^n(t) = const\,, \tag{21}$$

where $n \equiv 3(\gamma + 1)$. The constant parameter γ in the equation of state is usually restricted to the range $0 \leqq \gamma \leqq 1/3$ for the ordinary state but we formally extend to the range $-1 \leqq \gamma \leqq 1/3$ (negative pressure) here. This extension is relevant to unknown matter (dark-energy) suggested by the observed accelerating expansion of the universe [13]. The correspondent field equations of the Brans-Dicke theory are written down as follows [5], [8]:

$$2a\ddot{a} + \dot{a}^2 + 1 = -\frac{8\pi}{c^4}\frac{a^2 p}{\phi} - \frac{1}{2}\omega a^2\left(\frac{\dot{\phi}}{\phi}\right)^2 + a\dot{a}\left(\frac{\dot{\phi}}{\phi}\right) - \frac{8\pi}{(3+2\omega)c^4}\frac{a^2}{\phi}(\rho c^2 - 3p)\,, \tag{22}$$

$$\frac{3}{a^2}\left(\dot{a}^2+1\right) = \frac{16\pi(1+\omega)}{(3+2\omega)c^2}\frac{\rho}{\phi} + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\ddot{\phi}}{\phi} + \frac{24\pi}{(3+2\omega)c^4}\frac{p}{\phi}\,, \tag{23}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = \frac{8\pi}{(3+2\omega)c^2}\left(\rho - 3p/c^2\right)\,, \tag{24}$$

and the correspondent Machian solution is

$$\phi(t) = \frac{8\pi}{(3+2\omega)c^2}\zeta\rho(t)t^2\,, \tag{25}$$

$$a(t) = A(\omega)bt\,, \tag{26}$$

where some notations are

$$\zeta = 1/(\xi-2)\,, \tag{27}$$

$$\frac{3}{A^2(\omega)} = -(\frac{\omega}{2}+B)\,, \tag{28}$$

$$b = (4-\xi^2)^{-1/2}\,, \quad B = -3/(\xi^2-4)\,, \tag{29}$$

with a constraint $\omega/2 + B < 0$. The parameter ξ defined by $\xi \equiv 1 - 3\gamma$ may be restricted to the range $0 \leqq \xi < 2$, because of the continuity from the state $\xi = 1$ ($p = 0$) in the universe. We discard the range $2 < \xi \leqq 4$ here.

In this case, we obtain from Eq.(21)

$$\phi(t) \propto a^{2-n}(t)\,, \tag{30}$$

instead of Eq.(5) (the special case of $n = 3$), and thus

$$\frac{\dot{\phi}}{\phi} = -(2-n)\frac{\dot{a}}{a}\,. \tag{31}$$

Therefore, the correspondent equation to Eq.(13) becomes

$$2a\ddot{a} = -1 - \frac{(1+2\omega\gamma c^2)A^2(\omega)}{\xi+2} - \frac{1}{2}\left\{\omega(n-2)^2 + 2n - 2\right\}\dot{a}^2\,. \tag{32}$$

Through similar discussions, we obtain the acceleration rate of the perturbed term

$$\delta\ddot{a} = -\left\{\frac{1}{2}(\xi-2)^2\omega - \xi + 3\right\}\frac{\dot{a}}{a}\delta\dot{a} \tag{33}$$

for the perfect fluid case. We find the Machian cosmological model remains stable for an infinitesimal perturbation when the coupling constant satisfies

$$\omega > \omega_c \equiv \frac{2(\xi-3)}{(\xi-2)^2}. \tag{34}$$

We find $\omega > -3/2$ for $\xi = 0$, $\omega_c > -4$ for $\xi = 1$, and $\omega_c \to -\infty$ when $\xi \to 2$ respectively. For the closed model of the universe, because of the attractive gravitational force $G > 0$, the above Machian cosmological solution requests another constraint

$$\omega < -2, \quad \textit{for}\ 0 \leqq \xi \leqq 1 \tag{35}$$

or

$$\omega < \frac{6}{\xi^2-4}, \quad \textit{for}\ 1 < \xi < 2. \tag{36}$$

The inequality $\omega_c < 6/(\xi^2-4)$ always holds for the range $1 < \xi < 2$, and so for this range the Machian cosmological model is stable if

$$\omega_c < \omega < \frac{6}{\xi^2-4}. \tag{37}$$

For the range $(3-\sqrt{5})/2 < \xi \leqq 1$ the Machian cosmological model becomes stable when

$$\omega_c < \omega < -2, \tag{38}$$

but it is always unstable for an infinitesimal perturbation in case of $0 \leqq \xi \leqq (3-\sqrt{5})/2$, because of $\omega_c > -2$.

Finally we discuss the behavior of the Machian cosmological model in the generalized scalar-tensor theory of gravitation. We proposed as the possible variable coupling parameter

$$\omega(\phi) \equiv \frac{\omega_0}{\xi-2}, \tag{39}$$

where ω_0 is a constant ($\omega_0 > 3$), considering restrictions for the cosmological evolution in the framework of the Machian point of view[5], [9]. Our universe started from the Big Bang with $\xi = 0$ (the radiation era), passed the dust-dominated era ($\xi = 1$) in the early stage, and has been staying the negative pressure era ($\xi \approx 2$) for the almost all period. The barotropic state of the universe has been varying extremely slowly, and finally the universe will approach to the state of $\xi = 2$ as it expands for ever. As a result of the barotropic evolution of matter in the universe, the coupling function $\omega(\phi)$ of the Brans-Dicke scalar field diverges to the minus infinity and the gravitational constant $G(t)$ dynamically approaches to the constant G_∞.

We can determine whether the Machian universe is stable or not, in comparison between the coupling function $\omega(\phi)$ and the critical value ω_c. For an arbitrary ω_0 of the range $\omega_0 > 3$, the Machian model maintains stable, if the barotropic parameter ξ satisfies the following condition

$$\xi > \frac{2(\omega_0 - 3)}{\omega_0 - 2}. \tag{40}$$

Thus, in the neighborhood of the Big Bang with $\xi \approx 0$, where the universe does not satisfy this condition, the Machian cosmological model becomes unstable. Therefore, it occurs that a finite value of the parameter $\xi_c \equiv 2(\omega_0 - 3)/(\omega_0 - 2)$ exists and the stability of the universe reverses at this point. This situation is a little unnatural in physical meanings, and rather we consider that the unknown parameter ω_0 must be 3. In the particular case of $\omega_0 = 3$, the inequality

$$\omega(\phi) > \omega_c \tag{41}$$

identically holds for all $\xi > 0$, and the Machian universe evolves stably for the all period of time. In compensation for this situation, we need to recognize that the universe expands from the singular point of the Brans-Dicke scalar field with the coupling parameter $\omega = -3/2$. It is surely a defect that a singularity appears in the theory. However, let us remind the fact that the Big Bang itself is singular in the framework of classical theories. We think that it is remarkable that the unknown parameter ω_0 of the proposed coupling function $\omega(\phi)$ is consistently determined by means of discussions for the stability of the Machian cosmological model. In practical, even at the Big Bang, the universe does not completely consist of photon and so does not exactly start from the state of $\xi = 0$.

What does it mean that the Machian cosmological model is unstable? When an infinitesimal perturbation to the expansion parameter of the universe evolves exponentially, after all, the perturbation destroys the constraint like $a(t)\phi(t) = D$. The Machian constraint leads to the linearity of the expansion parameter and thus the deviation of this linearity results in the violation of its constraint. This means that the Machian cosmological model becomes no more

Machian. The asymptotic form $\phi = O(\rho/\omega)$ of the Barns-Dicke scalar field becomes to be destroyed and the Machian relation $G(t)M/c^2a(t) = const$ is also not maintained. Then the cosmological model seems to behave like an exact *general* solution of the field equations with the same energy-momentum tensor. It is trivial that we apply the above discussions to the Lemaitre universe or the Friedmann universe in General Relativity. Because the Lemaitre universe and the Friedmann universe are already general solutions of the Einstein's field equations for the homogeneous and isotropic universe with or without the cosmological constant respectively. An infinitesimal perturbation to the expansion parameter $a(t) + \delta a(t)$ itself again evolves as the general solution, the Lemaitre universe or the Friedmann universe.

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References

[1] C.Brans and R.H.Dicke, Phys. Rev. 124, 925 (1961).

[2] P.G.Bergmann, Int. J. Theor. Phys. 1, 25 (1968).

[3] R.V.Wagoner, Phys. Rev. D1, 3209 (1970).

[4] K.Nordtvedt, Astrophys. J. 161, 1059 (1970).

[5] Trends in Dark Matter Research, J.Val Blain(Ed.), Nova Science Publishers, Inc.(2005), p.141-p.189.

[6] A.Miyazaki, gr-qc/0012104, 2000.

[7] A.Miyazaki, gr-qc/0101112, 2001.

[8] A.Miyazaki, gr-qc/0102003, 2001.

[9] A.Miyazaki, gr-qc/0102105, 2001.

[10] A.Miyazaki, gr-qc/0103003, 2001.

[11] A.Miyazaki, gr-qc/0103013, 2001.

[12] X.Chen and M.Kamionkowski, Phys. Rev. D60, 104036 (1999).

[13] S.Perlmutter, M.S.Turner, and M.White, Phys. Rev. Lett. 83, 670 (1999).

[14] A.Miyazaki, Phys. Rev. Lett. 40, 725 (1978); 40, 1055(E) (1978).

[15] A.Miyazaki, Phys. Rev. D19, 2861 (1979); D23, 3085(E) (1981).

[16] H.Dehnen and O.Obregón, Astrophys. Space Sci. 14, 454 (1971).

[17] A.Miyazaki, Nuovo Cimento 68B, 126 (1982).