Behavior of the Brans-Dicke Scalar Field in Curved Space-Time

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Abstract

The behavior of the Brans-Dicke scalar field and the contribution of matter to the gravitational constant are discussed in some known exact cosmological solutions of the Brans-Dicke theory of gravitation for the homogeneous and isotropic universe. The sign of the contribution to the scalar field at the origin from matter in the universe inevitably reverses itself in our closed Machian cosmological model and hence the integrated contribution from whole matter of the universe amounts to the positive in spite of the negative coupling parameter ω. No inversion of the distance-dependence of contribution occurs in open space. It monotonously varies and converges to zero at infinity for small coefficients of the expansion parameter ($\alpha < \sqrt{2}$). In flat space, the contribution from the far region of the universe dampingly oscillates with distance, but the difference of signs between the contribution from nearby matter and from the whole universe does not arise.

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In recent papers, we discussed some cosmological models in the Brans-Dicke theory [1] of gravitation or in the generalized scalar-tensor theory of gravitation [2]-[4] in the framework of the Machian point of view, which is reduced to the postulate: *The scalar field of a proper cosmological solution has an asymptotic form* $\phi = O(\rho/\omega)$ *when the coupling parameter* ω *is large enough and converges to zero in the continuous limit* $\rho/\omega \to 0$ [5]. This postulate means that the gravitational constant, or inertia, is supported by matter in the universe and a cosmological model necessarily collapses when matter or its coupling to the scalar field vanishes, and seems to reflect original and intuitive ideas of Mach about inertia well. Moreover, this postulate is proved to be equivalent to the Machian relation $GM/c^2R = const$ for the homogeneous and isotropic universe [6].

By introducing *negative* pressure and *a varying cosmological constant* to our models, we found that a Machian solution in the generalized scalar-tensor theory of gravitation exhibits the slowly accelerating expansion of the universe [7], which is compatible with the recent measurements [8] of the distances to type Ia supernovae. Time-variation of the coupling function $\omega(\phi)$ in this model reveals why the measured coupling parameter ω of the Brans-Dicke theory of gravitation is so large at present [9], [10]. As the universe expands, the gravitational constant approaches dynamically to the constant G_∞ [10], which gives approximately the present value G_0 [11], and the cosmological constant decays rapidly [7]. Our Machian cosmological model finally becomes coincident with the Friedmann universe in General Relativity [7].

By introducing a scalar field φ as *dark-energy* we accomplished the scenario of the barotropic evolution of matter and our universe [12]. Our universe started from the Big Bang (in the classical meaning) with the coefficient of the equation of state $\gamma = 1/3$ (the radiation era), passed the dust-dominated era ($\gamma = 0$) rapidly in the early stage, and has been staying the negative pressure era ($\gamma \approx -1/3$) for the almost all period of its life $10^{10}yr$. The barotropic state of the universe has been varying extremely slowly from $\gamma = 1/3$ to $\gamma \approx -1/3$ and the universe will finally approach to the state of $\gamma = -1/3$ as it expands for ever.

This Machian solution of the (modified) generalized scalar-tensor theory of gravitation has fascinating aspects as a cosmological model for our universe, but it includes a crucial defect, the *negative* coupling constant ω in the theory. This means that the scalar field ϕ becomes a ghost with the negative energy-momentum. Brans and Dicke also insist that the coupling constant ω must be positive because of the positivity of contributions from matter to the scalar field ϕ [1]. However, this requirement does not necessarily seem to be true [13] in the Brans-Dicke theory of gravitation; There is a possibility of changing its sign for contributions of matter in the universe in curved space-time. In this paper we will systematically discuss the behavior of the Brans-Dicke scalar field in curved space-times, by means of known exact cosmological solutions in the Brans-Dicke theory, and contributions of matter in the universe to the gravitational constant, inertia, in order to clear the meaning of the negative coupling constant ω. The negative coupling constant is closely connected with the essence of our Machian cosmological models.

We start with the field equations of the Brans-Dicke theory of gravitation:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi}{c^4\phi}T_{\mu\nu} - \frac{\omega}{\phi^2}\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}\right) - \frac{1}{\phi}(\phi_{,\mu;\nu} - g_{\mu\nu}\Box\phi)\,, \tag{1}$$

$$\Box\phi = -\frac{8\pi}{(3+2\omega)c^4}T\,, \tag{2}$$

where $T_{\mu\nu}$ is the energy-momentum tensor, T is the contracted energy-momentum tensor, and $\Box$ denotes the generally-covariant d'Alembertian

$$\Box\phi \equiv \phi^{,\mu}{}_{;\mu} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)\phi\,. \tag{3}$$

The line element of the Friedmann-Robertson-Walker metric for the homogeneous and isotropic universe is described as

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + \sigma^2(\chi)(d\theta^2 + \sin^2\theta d\varphi^2)]\,, \tag{4}$$

where

$$\sigma(\chi) \equiv \begin{cases} \sin\chi & for\ k = +1\ (closed\ space) \\ \chi & for\ k = 0\ \ (flat\ space) \\ \sinh\chi & for\ k = -1\ \ (open\ space)\,. \end{cases} \tag{5}$$

The Brans-Dicke theory has a kind of particular solution for the homogeneous and isotropic universe with dust ($T = \rho c^2$), satisfying the relation $a(t)\phi(t) = const$ [13]-[15] (, see also [16]): For closed space with the coupling constant $\omega < -2$, the expansion parameter is

$$a(t) = [-2/(2+\omega)]^{1/2}\,t\,, \tag{6}$$

for open space with $\omega > -2$ ($\omega \neq -3/2$),

$$a(t) = [2/(2+\omega)]^{1/2}\,t\,, \tag{7}$$

and for flat space with $\omega = -2$,

$$a = a_0(t/t_0)\,, \tag{8}$$

respectively. The energy-momentum conservation gives

$$a^3(t)\rho(t) = const\,, \tag{9}$$

and the scalar field satisfies

$$\phi(t) = -[8\pi/(3+2\omega)c^2]\rho(t)t^2\,, \tag{10}$$

which yields the gravitational "constant" $G = (4+2\omega)/(3+2\omega)\phi$. Therefore, this model gives $G > 0$, $G = 0$, and $G < 0$ for $k = +1$, $k = 0$, and $k = -1$ respectively, and so only the closed model is physical in the usual meaning. However, our interest in this paper is to investigate the behavior of the Brans-Dicke scalar field for all phases of cosmological models. The scalar field ϕ itself remains physical for all phases of our models.

We consider a shell, the volume of which is restricted by the two hyper-surfaces $\chi = \chi_0$ and $\chi = \chi_1$, with $0 < \chi_0 < \chi_1 < \pi$ (or $+\infty$), in these models, and let the shell has additional mass density $\rho + \Delta\rho$. We will discuss effects to the scalar field by this shell in each curved space-time. The perturbation method used here is essentially the same as the discussions in Ref. [15]. The perturbation by the shell to the scalar field will depend on variables χ and t, and will be in general written as $\phi(t) + \Delta\phi(\chi, t)$ for the isotropic universe. The perturbed term $\Delta\phi(\chi, t)$ of the scalar field obeys the following partial differential equation derived from Eqs.(2) and (3) if the perturbation is small and the change of the metric is negligible:

$$\begin{aligned} & \frac{\partial^2}{\partial t^2}(\Delta\phi) + 3\frac{\dot a}{a}\frac{\partial}{\partial t}(\Delta\phi) - \frac{1}{a^2}\left[\frac{\partial^2}{\partial\chi^2}(\Delta\phi) + 2\lambda(\chi)\frac{\partial}{\partial\chi}(\Delta\phi)\right] \\ = & \frac{8\pi}{(3+2\omega)c^2}\Delta\rho \quad (\Delta\rho = 0 \textit{ outside the shell})\,, \end{aligned} \tag{11}$$

where

$$\lambda(\chi) \equiv \begin{cases} \cot\chi & for\ k = +1\ (closed\ space) \\ 1/\chi & for\ k = 0\ \ (flat\ space) \\ \coth\chi & for\ k = -1\ \ (open\ space) \end{cases} \tag{12}$$

and a dot denotes the usual partial derivative with respect to t. The homogeneous equation of Eq.(11) admits the separation with respect to the variables χ and t in each phase; let us write

$$\Delta\phi(\chi, t) = X(\chi)T(t)\,, \tag{13}$$

and then, denoting by S the separation constant, two equations arise:

$$\frac{d^2X}{d\chi^2} + 2\lambda(\chi)\frac{dX}{d\chi} - S\cdot X = 0\,, \tag{14}$$

$$\frac{d^2T}{dt^2} + 3\frac{\dot{a}(t)}{a(t)}\frac{dT}{dt} - \frac{S}{a(t)^2}T = 0 . \tag{15}$$

We obtain easily general solutions of Eq.(14) in each phase with $\lambda(\chi)$,

$$X(\chi) = \begin{cases} \left[C_1 \exp(-\sqrt{S-1}\chi) + C_2 \exp(\sqrt{S-1}\chi)\right] / \sin\chi & for\ S > 1 \\ (C_1 + C_2\chi)/\sin\chi & for\ S = 1 \\ \left[C_1 \cos(\sqrt{1-S}\chi) + C_2 \sin(\sqrt{1-S}\chi)\right] / \sin\chi & for\ S < 1 \end{cases} \tag{16}$$

for closed space,

$$X(\chi) = \begin{cases} \left[C_1 \exp(-\sqrt{S+1}\chi) + C_2 \exp(\sqrt{S+1}\chi)\right] / \sinh\chi & for\ S > -1 \\ (C_1 + C_2\chi)/\sinh\chi & for\ S = -1 \\ \left[C_1 \exp(-\sqrt{-S-1}\chi) + C_2 \exp(\sqrt{-S-1}\chi)\right] / \sinh\chi & for\ S < -1 \end{cases} \tag{17}$$

for open space, and

$$X(\chi) = \begin{cases} \left[C_1 \exp(-\sqrt{S}\chi) + C_2 \exp(\sqrt{S}\chi)\right] \chi^{-1} & for\ S > 0 \\ (C_1 + C_2\chi)\chi^{-1} & for\ S = 0 \\ \left[C_1 \cos(\sqrt{-S}\chi) + C_2 \sin(\sqrt{-S}\chi)\right] \chi^{-1} & for\ S < 0 \end{cases} \tag{18}$$

for flat space, where C_1 and C_2 are integral constants.

On account of the linear expansion of our cosmological models, $a(t) \equiv \alpha t$, the equation (15) reduces to

$$\frac{d^2T}{dt^2} + \frac{3}{t}\frac{dT}{dt} - \frac{S}{\alpha^2 t^2}T = 0 , \tag{19}$$

which has a general solution

$$T(t) = \begin{cases} C_1 t^{\mu-1} + C_2 t^{-\mu-1} & for\ S > -\alpha^2 \\ (C_1 + C_2 \ln t)t^{-1} & for\ S = -\alpha^2 \\ \left[C_1 \cos(\mu \ln t) + C_2 \sin(\mu \ln t)\right] t^{-1} & for\ S < -\alpha^2 , \end{cases} \tag{20}$$

where a constant $\mu \equiv \sqrt{|1 + S/\alpha^2|}$.

Taking the equation $\Delta\rho t^3 = \Delta\rho_0 t_0^3$ into account, we obtain in each phase as a particular solution of the inhomogeneous equation (11)

$$\Delta\phi_p(t) = -\frac{8\pi\Delta\rho_0 t_0^3}{(3+2\omega)c^2}\frac{1}{t} , \tag{21}$$

where an index 0 denotes a present value of each variable. As scalar fields in each region must be connected with each other smoothly for all t at the hypersurfaces $\chi = \chi_0$ and $\chi = \chi_1$, the time-dependence of the general solution of the homogeneous equation must be the same as that of the particular solution. Hence the unknown function $T(t)$ must vary with time as t^{-1}, and so the separation constant S is equal to $-\alpha^2$ for all phases $k = \pm 1, 0$.

By considering the separation constant $S = -\alpha^2 < 1$ in Eq.(16) and that a whole solution must be regular at $\chi = 0$ and $\chi = \pi$ for closed space, we construct an overall solution of the perturbed term $\Delta\phi(\chi, t)$ as

$$
\begin{aligned}
\Delta\phi_a(\chi,t) &= \frac{A}{t}\frac{\sin\kappa\chi}{\sin\chi} \quad (0 \leqslant \chi \leqslant \chi_0)\,, \\
\Delta\phi_b(\chi,t) &= \frac{1}{t}\left(\frac{B_1\cos\kappa\chi + B_2\sin\kappa\chi}{\sin\chi}\right) + \Delta\phi_p(t) \quad (\chi_0 \leqslant \chi \leqslant \chi_1)\,, \\
\Delta\phi_c(\chi,t) &= \frac{C}{t}\frac{\sin\kappa(\chi-\pi)}{\sin\chi} \quad (\chi_1 \leqslant \chi \leqslant \pi)\,,
\end{aligned}
\tag{22}
$$

where $\kappa \equiv \sqrt{1+\alpha^2}$, and A, B_1, B_2, and C are integral constants, which are determined by the conditions that solutions in each region must be connected with each other smoothly at $\chi = \chi_0$ and $\chi = \chi_1$, that is,

$$
\begin{aligned}
\Delta\phi_a(\chi_0,t) &= \Delta\phi_b(\chi_0,t)\,, \\
\Delta\phi_b(\chi_1,t) &= \Delta\phi_c(\chi_1,t)\,, \\
\partial_\chi\Delta\phi_a(\chi_0,t) &= \partial_\chi\Delta\phi_b(\chi_0,t)\,, \\
\partial_\chi\Delta\phi_b(\chi_1,t) &= \partial_\chi\Delta\phi_c(\chi_1,t)\,.
\end{aligned}
\tag{23}
$$

After solving Eqs.(23) with Eqs.(22) simultaneously for A, B_1, B_2, and C, we get the perturbed scalar field at the origin

$$
\begin{aligned}
\Delta\phi_0(t;\chi_0,\chi_1) &= \lim_{\chi\to 0}\Delta\phi_a(\chi,t) \qquad (24) \\
&= -\frac{8\pi}{(3+2\omega)c^2}\frac{\Delta\rho t^2}{\sin\kappa\pi}\left[\kappa\sin\chi\cos\kappa(\chi-\pi) - \cos\chi\sin\kappa(\chi-\pi)\right]_{\chi_0}^{\chi_1}\,,
\end{aligned}
$$

or in an integral form

$$
\Delta\phi_0(t;\chi_0,\chi_1) = -\frac{8\pi\Delta\rho t^2}{(3+2\omega)c^2}\frac{\alpha^2}{\sin\kappa\pi}\int_{\chi_0}^{\chi_1}\sin\chi\sin\kappa(\chi-\pi)d\chi\,. \tag{25}
$$

When the shell is thin enough, on account of a theorem of mean value, by representing M_s mass of the shell with density $\Delta\rho$ in closed space

$$
M_s = 4\pi\Delta\rho a^3\int_{\chi_0}^{\chi_1}\sin^2\chi d\chi\,, \tag{26}
$$

we obtain the distance-dependence of contribution to the scalar field at the origin from matter of the universe with mass M_s at the point $\chi = \chi^*$ in the background metric (4) with $\sigma(\chi) = \sin\chi$ for closed space

$$\Delta\phi_0(t\,;\chi^*) = -\frac{2M_s}{(3+2\omega)c^2\alpha t}\frac{\sin\kappa(\chi^*-\pi)}{\sin\kappa\pi\sin\chi^*}\,. \tag{27}$$

A significant feature of this distance-dependence is an inversion of the sign of contribution at a point $\chi_c = (1-1/\kappa)\pi$. The contribution to $\Delta\phi_0$ from matter in the region inside of χ_c is negative when $\omega < -2$ and $\kappa > 1$, and the contribution from the region outside of χ_c becomes positive. The contribution from nearby matter obeys the inverse proportion to the proper distance ($\Delta\phi_0 \propto -1/r$), and the contribution from distant matter oscillates slowly with distance when the constant κ becomes large. However, the contribution from the whole universe, given by $\Delta\phi_0(t\,;0,\pi)$, always amounts to the positive even if $3+2\omega < 0$ in closed space. This fact gives a counter example to the assertion of Brans and Dicke [1], and does not seem to be known well yet. We exhibit here two graphs of the distance-dependence of contribution of matter; Figure 1 represents the contribution from matter with mass M_s at the point $\chi = \chi^*$ and Figure 2 represents the sliced contribution by the thin shell of the universe at $\chi = \chi^*$ (multiplied the former by $\sin^2\chi^*$). The solid line is for $\alpha = 1$, the dotted line for $\alpha = 2$, and is described by an arbitrary scale in y-axis, respectively.

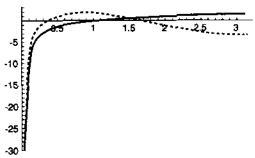

Figure 1.

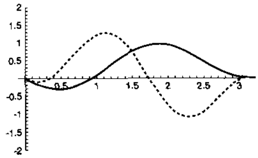

Figure 2.

In the case of open space with the separation constant $S = -\alpha^2 > -1$, we build from Eq.(17) for the perturbed scalar field

$$\begin{aligned}
\Delta\phi_a(\chi,t) &= \frac{A}{t}\frac{\sinh\kappa\chi}{\sinh\chi} \quad (0 \leqslant \chi \leqslant \chi_0)\,,\\
\Delta\phi_b(\chi,t) &= \frac{1}{t}\left(\frac{B_1\cosh\kappa\chi + B_2\sinh\kappa\chi}{\sinh\chi}\right) - \Delta\phi_p(t) \quad (\chi_0 \leqslant \chi \leqslant \chi_1)\,,\\
\Delta\phi_c(\chi,t) &= \frac{C}{t}\frac{\cosh\kappa\chi}{\sinh\chi} \quad (\chi_1 \leqslant \chi < +\infty)\,,
\end{aligned} \tag{28}$$

where $\kappa \equiv \sqrt{1-\alpha^2}$, after the discussions on the behavior of the scalar field at the origin or at infinity. Remark that only a minus sign of $\Delta\phi_p$ can give a physical solution for the scalar field at the origin $\Delta\phi_0$. By solving the correspondent equations (23) simultaneously and after the similar calculations, on account of mass of the shell in open space

$$M_s = 4\pi\Delta\rho a^3 \int_{\chi_0}^{\chi_1} \sinh^2\chi d\chi\,, \tag{29}$$

the distance-dependence of contribution to the scalar field at the origin from matter at $\chi = \chi^*$ in the background metric (4) with $\sigma(\chi) = \sinh\chi$ for open space is derived:

$$\Delta\phi_0(t\,;\chi^*) = -\frac{2M_s}{(3+2\omega)c^2\alpha t}\frac{\cosh\kappa\chi^*}{\sinh\chi^*}\,. \tag{30}$$

No inversion of the contribution from overall matter in the universe happens for all coefficients of the expansion parameter α $(0 < \alpha < 1)$ in this case.

For open space with $S = -1$ $(\alpha = 1)$, one finds by means of the similar way for the perturbed scalar field

$$\begin{aligned}
\Delta\phi_a(\chi, t) &= \frac{A}{t}\frac{\chi}{\sinh\chi} \quad (0 \leqslant \chi \leqslant \chi_0)\,,\\
\Delta\phi_b(\chi, t) &= \frac{1}{t}\left(\frac{B_1 + B_2\chi}{\sinh\chi}\right) - \Delta\phi_p(t) \quad (\chi_0 \leqslant \chi \leqslant \chi_1)\,,\\
\Delta\phi_c(\chi, t) &= \frac{C}{t}\frac{1}{\sinh\chi} \quad (\chi_1 \leqslant \chi < +\infty)\,,
\end{aligned} \tag{31}$$

and the distance-dependence of contribution

$$\Delta\phi_0(t\,;\chi^*) = -\frac{2M_s}{(3+2\omega)c^2 t}\frac{1}{\sinh\chi^*}\,. \tag{32}$$

For open space with $S < -1$ $(\alpha > 1)$, one also finds

$$\begin{aligned}
\Delta\phi_a(\chi, t) &= \frac{A}{t}\frac{\sinh\kappa\chi}{\sinh\chi} \quad (0 \leqslant \chi \leqslant \chi_0)\,,\\
\Delta\phi_b(\chi, t) &= \frac{1}{t}\left(\frac{B_1\cosh\kappa\chi + B_2\sinh\kappa\chi}{\sinh\chi}\right) + \Delta\phi_p(t) \quad (\chi_0 \leqslant \chi \leqslant \chi_1)\,,\\
\Delta\phi_c(\chi, t) &= \frac{C}{t}\frac{\cosh\kappa\chi}{\sinh\chi} \quad (\chi_1 \leqslant \chi < +\infty)\,,
\end{aligned} \tag{33}$$

where $\kappa \equiv \sqrt{\alpha^2 - 1}$, and obtains similarly

$$\Delta\phi_0(t\,;\chi^*) = -\frac{2M_s(\alpha^2-2)}{(3+2\omega)c^2\alpha^3 t}\frac{\cosh\kappa\chi^*}{\sinh\chi^*}\,. \tag{34}$$

Thus, for all cases of the separation constant S, any inversion of the contribution never occur in open space. The contribution of matter in the universe monotonously varies and converges to zero at infinity for $\kappa < 1$.

In the case of flat space with the separation constant $S = -\alpha^2 < 0$, we construct from Eq.(18) an overall solution for the perturbed scalar field

$$\begin{aligned}
\Delta\phi_a(\chi, t) &= \frac{A}{t}\frac{\sin\kappa\chi}{\chi} \quad (0 \leqslant \chi \leqslant \chi_0)\,, \\
\Delta\phi_b(\chi, t) &= \frac{1}{t}\left(\frac{B_1\cos\kappa\chi + B_2\sin\kappa\chi}{\chi}\right) + \Delta\phi_p(t) \quad (\chi_0 \leqslant \chi \leqslant \chi_1)\,, \\
\Delta\phi_c(\chi, t) &= \frac{C}{t}\frac{\cos\kappa\chi}{\chi} \quad (\chi_1 \leqslant \chi < +\infty)\,, \qquad (35)
\end{aligned}$$

where $\kappa \equiv \alpha$. Remark that a trivial solution arises for $\sin\kappa\chi/\chi$ in the C-region $(\chi_1 \leqslant \chi < +\infty)$. After considering mass of the shell with density $\Delta\rho$ in flat space

$$M_s = 4\pi\Delta\rho a^3 \int_{\chi_0}^{\chi_1} \chi^2 d\chi\,, \qquad (36)$$

we get the distance-dependence of contribution to the scalar field at the origin from matter at $\chi = \chi^*$ in the flat background metric (4) with $\sigma(\chi) = \chi$

$$\Delta\phi_0(t\,;\chi^*) = \frac{2M_s}{(3+2\omega)c^2\alpha t}\frac{\cos\kappa\chi^*}{\chi^*}\,. \qquad (37)$$

It is natural that the distance-dependence with $\omega = -2$ surely gives $\Delta\phi_0 \propto -1/r$ for the contribution from nearby matter, but it is somewhat peculiar that it behaves like a damping oscillator in the far region of the universe in spite of the flat space. However, the sign of the contribution from the whole universe $\Delta\phi_0(t\,;0,+\infty)$, or the cosmological solution itself, is coincident with that of the contribution from nearby matter. Thus, we confirm that it is only the closed space in which the difference of signs between the contribution from nearby matter and from the whole universe arises. Physical meaning of the oscillation of the contribution in flat space remains unknown.

We next survey the behavior of the scalar field in other exact cosmological models. We know two examples of such exact cosmological solutions: the Machian flat solution [6] with the arbitrary coupling constant ω ($\omega \neq -3/2$. The condition $\omega < -2$ yields an attractive gravitational force $G > 0$.)

$$a = a_0(t/t_0)^2\,, \qquad (38)$$

$$\phi(t) = -[2\pi/(3+2\omega)c^2]\rho(t)t^2\,, \qquad (39)$$

and the Brans-Dicke flat solution [1] for the homogeneous and isotropic universe

$$\phi = \phi_0 (t/t_0)^r \, , \quad a = a_0 (t/t_0)^q \tag{40}$$

with

$$r = 2/(4+3\omega) \, , \quad q = (2+2\omega)/(4+3\omega) \, . \tag{41}$$

The correspondent equation to determine the time-dependence of the contribution in these models, derived from Eq.(15), is reduced to the following general form:

$$x^2 \frac{d^2 y}{dx^2} + ax \frac{dy}{dx} + (bx^m + c) y = 0 \, . \tag{42}$$

This differential equation has a general solution (see [17]) when $m \neq 0$ and $b \neq 0$

$$y = x^{\frac{1-a}{2}} Z_\nu \left(\frac{2}{m} \sqrt{b} x^{\frac{m}{2}} \right) , \qquad with \ \nu \equiv \frac{1}{m} \sqrt{(1-a)^2 - 4c} \, , \tag{43}$$

where

$$Z_\nu(x) \equiv C_1 J_\nu(x) + C_2 Y_\nu(x) \, , \tag{44}$$

and $J_\nu(x)$ are Bessel functions of the first kind, $Y_\nu(x)$ are Bessel functions of the second kind respectively. We find that these solutions never give a function of powers, the time-dependence of a particular solution of the correspondent inhomogeneous equation (11). This difficulty means that no separable solutions with respect to χ and t for the contribution from matter in these cosmological models and suggests for us to seek other methods to determine the behavior of the scalar field in curved space-times.

Let us suppose a static point-like source described by a delta function at the origin in any cosmological models expanding arbitrarily for the homogeneous and isotropic universe. No explicit time-dependence of the contribution from the point mass appears and we may put the separation constant $S = 0$ in Eq.(14):

$$\frac{d^2 X}{d\chi^2} + 2\lambda(\chi) \frac{dX}{d\chi} = 0 \, , \tag{45}$$

which yields a general solution

$$X(\chi) = \begin{cases} C_1 + C_2 \cot \chi & for \ k = +1 \ (closed \ space) \\ C_1 + C_2 \chi^{-1} & for \ k = 0 \quad (flat \ space) \\ C_1 + C_2 \coth \chi & for \ k = -1 \ \ (open \ space) \, . \end{cases} \tag{46}$$

On account of the condition that any solutions behave like $1/\chi$ in the neighborhood of the origin, we select as physical solutions $\cot \chi$, χ^{-1}, and $\coth \chi$ for $k = +1$, $k = 0$, and $k = -1$ respectively. Remark $\lim_{\chi \to \infty} \coth \chi = 1$ for open space, which means the contribution from matter at the origin does not vanish even at infinity. The solution $\cot\chi$ for closed space is antisymmetric at $\chi = \pi/2$ and diverges at the opposite side to the origin of the universe, $\chi = \pi$. These behaviors of the contribution from the static point mass at the origin are very different from those of the contribution from the spherical symmetric thin shell at $\chi = \chi^*$ in the expanding universe. A value of the separation constant S, which depends on how the universe expands, plays an important role in the present problems.

For more general discussions, we need to investigate Green's function in curved space-time for the generally-covariant d'Alembertian. The methods by means of Green's function in curved space-time enable us to take a global view of the contribution of matter to the scalar field and may make clear the causal aspects of Mach's principle with retarded or advanced effects. Seeliger's paradox for the scalar field in infinite space also appears in our perturbation methods by integrating the contribution for the whole universe in open or flat space. (Cosmological solutions itself do not surely include this paradox.) The problems of Green's functions in curved space-times remain in future papers.

Our most interesting concern is whether the inconsistency of signs between the contribution from nearby matter and from the whole universe generally occurs in closed space. Is it a special phenomenon only in our closed Machian cosmological model? If so, the gravitational constant, or inertia, is dominated by matter in the far region of the universe. Is the linear expansion of our cosmological model, which means all observers fixed to matter in the universe do not accelerate to each other, essential in the Machian point of view or in problems of behavior of the scalar field? We, unfortunately, have not known any other exact cosmological solutions of the Brans-Dicke field equations for closed space yet. If the above propositions are affirmatively proved, they will make our Machian cosmological models more advantageous. Our universe is inevitably restricted to closed space because of giving an attractive gravitational force $G > 0$.

We restricted our discussions here for the behavior of the scalar field in the universe to the Brans-Dicke theory of gravitation. The Machian cosmological model with the cosmological constant and negative pressure in the generalized scalar-tensor theory of gravitation gives the essentially same linear expansion parameter $a(t) = \alpha(\omega)t$ as in the Brans-Dicke theory. Its coefficient $\alpha(\omega)$ varies quasi-statically with the coupling constant ω according to the evolution of the universe and realizes substantially the accelerated expansion of the universe at present. Therefore, the similar calculations for the contribution of matter in the universe are possible and the same inversion of signs between the contribution from nearby matter and the integrated contribution from the whole universe also appears in closed space in the framework of the generalized scalar-tensor theory of gravitation.

More fundamental problems concerned with a negative coupling, that is, a ghost or the negative energy-momentum of the field, still remain to be solved for consistency of our Machian point of view.

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