Uniqueness of the Machian Cosmological Solution in the Brans-Dicke Theory

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Abstract

Cosmological solutions with Machian conditon $\phi=O(\rho/\omega)$ are generally explored for the homogeneous and isotropic universe with dust matter in the Brans-Dicke theory. It is proved that the Machian solution with arbitrary values of ω is unique for the flat, closed ($\omega<-2$), and open ($\omega>-2$) spaces respectively and its scalar field is restricted to the form $\phi\propto\rho t^2$. This form of the scalar field leads to the well-known Machain relation $GM/c^2R\sim1$. We find that this Machian relation and the asymptotic behavior $\phi=O(\rho/\omega)$ of the scalar field for a large value of ω are equivalent to each other in the Brans-Dicke theory. The criterion $\phi=O(\rho/\omega)$ would play important roles as a guiding principle to investigate cosmological problems in the Machian point of view.

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I Introduction

In the preceding paper [1], we discussed the asymptotic behavior of the

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scalar field ϕ of the Brans-Dicke theory [2] for the large enough coupling parameter ω, and proposed two postulates in the Machian point of view: *The scalar field of a proper cosmological solution should have the asymptotic form* $\phi = O(\rho/\omega)$ *and should converge to zero in the continuous limit* $\rho/\omega \to 0$. (Let us call it "the Machian solution" here.) *The scalar field by locally-distributed matter should exhibit the asymptotic behavior* $\phi = \langle\phi\rangle + O(1/\omega)$. When we consider local problems, for example black holes, in the Brans-Dicke theory, we always premise tacitly the presence of cosmologically-distributed matter in the whole universe. If we restrict to the scalar field with the asymptotic behavior $\phi = \langle\phi\rangle + O(1/\omega)$, we can discuss possible solutions for locally -distributed matter with an asymptotically-flat boundary condition for the metric tensor $g_{\mu\nu}$ and with another boundary condition that the local scalar field ϕ converges to $\langle\phi\rangle$, which is given by the experimental gravitational constant G, at the asymptotic region $(r \to +\infty)$. We may forget our cosmological environment on this assumption.

Taking the large coupling parameter ω restricted by the recent measurements [3] ($|\omega| \gtrsim 10^3$) into account, there exists only an extremely small difference between general relativity and the Brans-Dicke theory for local problems. However, there appears a crucial difference for cosmological problems, especially in the Machian point of view. It becomes more important to survey systematically such Machian solutions in the Brans-Dicke theory. What are general cosmological solutions with Machian condition $\phi = O(\rho/\omega)$ in the Brans-Dicke theory? The purpose of this article is seeking the general existence of solutions satisfying asymptotically $\phi = O(\rho/\omega)$ for a large value of ω in the Brans-Dicke theory. We restrict our discussions to the homogeneous and isotropic universe with perfect fluid matter (with negligible pressure) here.

Such an example is already known [4] and [5], [6] for the closed model, and [7] for the flat and the open models. The scalar field ϕ of this cosmological model has the following form for all kind of spaces:

$$\phi(t) = -[8\pi/(3+2\omega)c^2]\rho(t)t^2. \tag{1}$$

This particular solution was obtained by adding the constraint $a\phi = const$, which means $GM/c^2a = const$ (, where a is the expansion parameter). The closed cosmological model exists only for a particular value [6]

$$\frac{G(t)M}{c^2a(t)} = \pi, \tag{2}$$

where M is a total mass of the closed universe. Therefore, the well-known Machian relation

$$\frac{GM}{c^2R} \sim 1, \tag{3}$$

where R is a radius of the universe, is automatically satisfied in this cosmological model *all the time*. However, this relation is ambiguous as to the meaning of the radius R or the mass M of the universe, and might not be essential to discuss Machian cosmological models.

As the more fundamental Machian relation in the Brans-Dicke theory, according to the postulate in the preceding paper [1], let us require the following relation of the scalar field

$$\phi = O(\rho/\omega). \tag{4}$$

When the mass density ρ of the universe goes to zero or the coupling between the scalar field and matter vanishes ($|\omega| \to \infty$), the scalar field ϕ also converges to zero. This means that a particle in any other empty space does not have the inertial properties. A direct form of this scalar field ϕ is given as

$$\phi(t)=\frac{8\pi\rho(t)}{(3+2\omega)c^2}\Psi(t), \tag{5}$$

where Ψ is a function of t and may depend on ω as

$$\Psi(t)=\Psi_0(t)+O(1/\omega). \tag{6}$$

We, however, find that it is rather complicate to assume this type of solutions and seek unknown functions Ψ. Let us start simply from another relation

$$\phi(t)=\frac{8\pi}{(3+2\omega)c^2}\Phi(t). \tag{7}$$

An unknown function Φ may also have the same dependence of ω as Eq. (6).

II Basic Equations and the Flat-Space Case

For the homogeneous and isotropic universe, the nonvanishing component of the energy-momentum tensor of the perfect-fluid with negligible pressure is $T_{00}=-\rho c^2$, and the contracted energy-momentum tensor is $T=\rho c^2$. Thus the nonvanishing field equations we need solve simultaneously in the Brans-Dicke theory are

$$2a\ddot{a}+\dot{a}^2+k=-\frac{8\pi}{(3+2\omega)c^2}\frac{a^2\rho}{\phi}-\frac{1}{2}\omega a^2\left(\frac{\dot{\phi}}{\phi}\right)^2+a\dot{a}\left(\frac{\dot{\phi}}{\phi}\right), \tag{8}$$

$$\frac{3}{a^2}(\dot{a}^2+k)=\frac{16\pi(1+\omega)}{(3+2\omega)c^2}\frac{\rho}{\phi}+\frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2+\frac{\ddot{\phi}}{\phi}, \tag{9}$$

$$\ddot{\phi}+3\frac{\dot{a}}{a}\dot{\phi}=\frac{8\pi}{(3+2\omega)c^2}\rho, \tag{10}$$

where a dot denotes the partial derivative with respect to t and $k=+1$, $k=0$, and $k=-1$ for closed, flat, and open spaces, respectively. However, we can use the conservation law of the energy-momentum, which is derived from the field equations, as the independent equation instead of Eq.(8), and obtain for this case

$$2\pi^2 a^3(t)\rho(t)=M=const. \tag{11}$$

If we substitute Eq.(7) into Eq.(10), we get directly

$$\ddot{\Phi}+3\frac{\dot{a}}{a}\dot{\Phi}=\rho. \tag{12}$$

Let us suppose the mass density ρ and the function Φ do not depend on the coupling parameter ω, then we observe that the ratio $\dot{a}/a$ also does not depend ω:

$$\frac{\dot{a}}{a}\equiv\gamma(t), \tag{13}$$

where a function γ depends on only t. By integrating Eq.(13), we find the expansion parameter should have a form

$$a(t)\equiv A(\omega)\alpha(t), \tag{14}$$

where A and α are arbitrary functions of only ω and t respectively.

From Eqs.(10) and (11) after integration, we obtain

$$\dot{\phi}a^3=[4M/(3+2\omega)\pi c^2]t, \tag{15}$$

where we take a constant of integration into zero. From Eqs. (9), (10), (11), and (15), we get

$$\left[\left(\frac{\dot{a}}{a}\right)+\frac{1}{2}\left(\frac{\dot{\phi}}{\phi}\right)\right]^2+\frac{k}{a^2}=\frac{1}{4}\left(1+\frac{2\omega}{3}\right)\left[\left(\frac{\dot{\phi}}{\phi}\right)^2+4\left(\frac{\dot{\phi}}{\phi}\right)\frac{1}{t}\right]. \tag{16}$$

Similarly we obtain

$$\dot{\Phi}a^3=(M/2\pi^2)t \tag{17}$$

and

$$\left[\left(\frac{\dot{a}}{a}\right)+\frac{1}{2}\left(\frac{\dot{\Phi}}{\Phi}\right)\right]^2+\frac{k}{a^2}=\frac{1}{4}\left(1+\frac{2\omega}{3}\right)\left[\left(\frac{\dot{\Phi}}{\Phi}\right)^2+4\left(\frac{\dot{\Phi}}{\Phi}\right)\frac{1}{t}\right]. \tag{18}$$

It should be noted that the scalar field ϕ appears only as the ratio $\dot{\phi}/\phi$ in Eq.(16).

We know the general exact solution of Eqs. (15) and (16) for the flat-space case [2], and this solution is also valid for the early expansion phases for the closed or the open spaces. Therefore, Eqs. (17) and (18) has the similar general solution in the same situation:

$$\Phi=\Phi_0(t/t_0)^r,\ r=2/(4+3\omega), \tag{19}$$

$$\Phi_0=[(4+3\omega)/2]\rho_0 t_0^2. \tag{20}$$

In the result, Eq. (7) has become the usual non-Machian cosmological solution the asymptotic behavior of which is

$$\phi=\langle\phi\rangle+O(1/\omega). \tag{21}$$

Thus we conclude that the function Φ should not depend on the coupling parameter ω for the Machian cosmological solution. Equation (7) with the function $\Phi(t)$ depending on only t is the most general form for the Machian solution $\phi=O(\rho/\omega)$. The mass density $\rho(t)$ is given first and should not also depend on ω. So we confirm that the expansion parameter $a(t)$ is described like Eq.(14). The ω-dependence of the expansion parameter Eq. (14) is put into the total mass of the universe $M(\omega)$. Finally we obtain from Eqs. (9), (10), (7), and (14)

$$\frac{\omega}{2}\left[\left(\frac{\dot{\Phi}}{\Phi}\right)^2+\frac{4\rho}{\Phi}\right]-\frac{3k}{A^2(\omega)a^2}=3\left(\frac{\dot{a}}{a}\right)^2+3\left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{\Phi}}{\Phi}\right)-\frac{3\rho}{\Phi}. \tag{22}$$

The independent equations which we solve simultaneously are Eqs. (22), (17), and (11) for $k=0,\ \pm1$.

Let us discuss the flat model ($k=0$) first. If we request that Eq. (22) is always satisfied for all arbitrary values of ω, we find the two following conditions must be held identically,

$$\left(\frac{\dot{\Phi}}{\Phi}\right)^2+\frac{4\rho}{\Phi}\equiv0, \tag{23}$$

and

$$\left(\frac{\dot{a}}{a}\right)^2+\left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{\Phi}}{\Phi}\right)-\frac{\rho}{\Phi}\equiv0. \tag{24}$$

After eliminating the term ρ/Φ from Eqs. (23) and (24), we can resolve the result into factors and obtain

$$\left[\left(\frac{\dot{\Phi}}{\Phi}\right)^2+2\left(\frac{\dot{a}}{a}\right)\right]^2=0. \tag{25}$$

The integral of the term in the middle bracket gives

$$\Phi(t)a^2(t)=const. \tag{26}$$

On the other hand, we get from Eqs. (23), (17), and (11)

$$\Phi(t)=-\frac{M}{8\pi^2A^3}\frac{t^2}{a^3(t)}\,. \tag{27}$$

If we eliminate the term a^3 from Eq. (27) using Eq. (17), we obtain simply

$$\frac{\dot{\Phi}}{\Phi}=-\frac{4}{t}\,. \tag{28}$$

We can integrate this equation exactly and find

$$\Phi(t)\propto t^{-4}, \tag{29}$$

Substituting this equation into Eq. (27) yields

$$a(t)\propto t^2. \tag{30}$$

Though time-dependences of $\Phi(t)$ and $a(t)$ are compatible with Eq. (26), we can not determine the coefficient of $a(t)$. Equations (23), (24), and (12) are no more independent to cach other and the solution becomes indefinite. It is also obvious that the coefficient of the expansion parameter $a(t)$ becomes indefinite because only the ratio $\dot{a}/a$ appears in Eqs. (12) and (18) for the flat-space case $(k=0)$. The coefficient of $a(t)$ is also indefinite in the particular solution with $\omega=-2$ for the flat space. It seems that this indefiniteness is the situation characteristic of the flat -space case.

If we fix a value a_0 of the expansion parameter at an arbitrary time t_0, we may write down the indefinite solution as

$$a(t)=a_0(t/t_0)^2 \tag{31}$$

and the scalar field $\phi(t)$ is given by the mass density $\rho(t)$ as

$$\phi(t)=-[2\pi/(3+2\omega)c^2]\rho(t)t^2, \tag{32}$$

which is valid for all arbitrary values $(\omega\neq-3/2)$ of the coupling parameter ω ($\omega<-2$ for $G>0$). If we restrict to the causally-related region, we observe the following relation *for all t,* taking the radius of the horizon $R(t)=t$ and the total mass inside the horizon $M(t)=(4/3)\pi R^3(t)\rho(t)$ into account:

$$\frac{G(t)M(t)}{c^2R(t)}=-4(2+\omega)/3=const. \tag{33}$$

It is essential for the relation $GM/c^2R=const$ that the scalar field ϕ has the form $\rho(t)t^2$, which is derived from Eq. (23).

III Closed-Space Case

For the closed or the open models ($k=\pm1$), we obtain from Eq. (22) the two constraints:

$$\frac{\omega}{2}\left[\left(\frac{\dot{\Phi}}{\Phi}\right)^2+\frac{4\rho}{\Phi}\right]-\frac{3k}{A^2(\omega)a^2}\equiv C(t), \tag{34}$$

$$3\left(\frac{\dot{a}}{a}\right)^2+3\left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{\Phi}}{\Phi}\right)-\frac{3\rho}{\Phi}\equiv C(t), \tag{35}$$

where C is a function of only t. In order that Eq. (34) is satisfied for all arbitrary values of ω, the function A must have the following form

$$\frac{3}{A^2(\omega)}=\left|\frac{\omega}{2}+B\right|, \tag{36}$$

where B is a constant with no dependence of ω. Therefore, in the Machian solutions for the homogeneous and isotropic universe (except the flat space), the expansion parameter $a(t)$ exhibits generally the asymptotic behavior $O(1/\sqrt{|\omega|})$ for the large enough coupling parameter ω.

For $k=+1$, and $\omega/2+B<0$, Eq. (36) gives

$$\frac{3}{A^2(\omega)}=-\left(\frac{\omega}{2}+B\right), \tag{37}$$

and we get from Eq. (22)

$$\frac{\omega}{2}\left[\left(\frac{\dot{\Phi}}{\Phi}\right)^2+\frac{4\rho}{\Phi}+\frac{1}{a^2}\right]=3\left(\frac{\dot{a}}{a}\right)^2+3\left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{\Phi}}{\Phi}\right)-\frac{3\rho}{\Phi}-\frac{B}{a^2}. \tag{38}$$

We need to hold the two following identities to satisfy this equation for all arbitrary ω:

$$\left(\frac{\dot{\Phi}}{\Phi}\right)^2+\frac{4\rho}{\Phi}\equiv-\frac{1}{a^2}, \tag{39}$$

$$3\left(\frac{\dot{a}}{a}\right)^2+3\left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{\Phi}}{\Phi}\right)-\frac{3\rho}{\Phi}\equiv\frac{B}{a^2}. \tag{40}$$

If we eliminate the derivative $\dot{\Phi}$ from Eq. (39) using Eq. (17), we obtain

$$\Phi^2(t)+\frac{2M}{\pi^2A^3a(t)}\Phi(t)+\left(\frac{Mt}{2\pi^2A^3a^2(t)}\right)^2=0, \tag{41}$$

and find directiy

$$\Phi(t)=-\frac{M}{\pi^2A^3a(t)}(1\pm\sqrt{1-t^2/4a^2(t)}). \tag{42}$$

In order that the function Φ is real for all times $(0\leqq t<+\infty)$, the time-dependence of the expansion parameter must be at least linear:

$$a(t)\propto t. \tag{43}$$

Therefore, we observe from Eq. (42)

$$\Phi(t)a(t)=const, \tag{44}$$

and hence

$$\Phi(t)\propto t^{-1}. \tag{45}$$

We can determine the expansion parameter $a^2(t)=-[2/(\omega+2)]t$ directly from Eq. (18) and then find easily the coefficient of $a(t)$ and the value of B satisfying Eq. (40):

$$a(t)=t/\sqrt{3},\ B=1, \tag{46}$$

and we get

$$\Phi(t)a(t)=-\frac{3M}{2\pi^2A^3}, \tag{47}$$

which means

$$\Phi(t) = -\rho(t)t^2. \tag{48}$$

This is nothing but the Machian solution Eq. (1) for the closed space ($k = +1$) .

Next we discuss the possibility of the bounded universe with a finite time. We restrict our discussions to the early expansion phases, and let us write

$$a(t) = bt^{\beta}, \tag{49}$$

where b and β are constants. When $\beta > 1$, the second term of Eq. (42)

$$\sqrt{1 - 1/4b^2 t^{2(\beta-1)}} \tag{50}$$

diverges at $t = 0$, and there exist no solutions in this neighborhood. When $\beta < 1$, for early expansion phases, the relation

$$a(t) \gg t \tag{51}$$

is valid, and we obtain from Eq. (42)

$$\Phi(t) = -\frac{2M}{\pi^2 A^3 a(t)}, \ or\ 0, \tag{52}$$

which does not satisfy Eq. (40). Thus, when $\beta \neq 1$, the Machian solution does not exist for a finite and meaningful time in this case ($k = +1$, $\omega/2 + B < 0$).

For $k = +1$, and $\omega/2 + B > 0$, the coefficient $A(\omega)$ has the form

$$\frac{3}{A^2(\omega)} = \frac{\omega}{2} + B, \tag{53}$$

and Eq. (22) yields

$$\frac{\omega}{2}\left[\left(\frac{\dot{\Phi}}{\Phi}\right)^2 + \frac{4\rho}{\Phi} - \frac{1}{a^2}\right] = 3\left(\frac{\dot{a}}{a}\right)^2 + 3\left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{\Phi}}{\Phi}\right) - \frac{3\rho}{\Phi} + \frac{B}{a^2}, \tag{54}$$

which produces, through the similar discussions for all arbitrary ω, the two following identities:

$$\left(\frac{\dot{\Phi}}{\Phi}\right)^2 + \frac{4\rho}{\Phi} \equiv \frac{1}{a^2}, \tag{55}$$

$$3\left(\frac{\dot{a}}{a}\right)^2+3\left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{\Phi}}{\Phi}\right)-\frac{3\rho}{\Phi}\equiv-\frac{B}{a^2}\,. \tag{56}$$

The elimination of the derivative $\dot{\Phi}$ from Eq. (55) by Eq. (17) yields

$$\Phi^2(t)-\frac{2M}{\pi^2A^3a(t)}\Phi(t)-\left(\frac{Mt}{2\pi^2A^3a^2(t)}\right)^2=0, \tag{57}$$

which is solved directly to

$$\Phi(t)=\frac{M}{\pi^2A^3a(t)}\left(1\pm\sqrt{1+t^2/4a^2(t)}\right). \tag{58}$$

It is enough to indicate that no solutions exist in the neighborhood of $t=0$ in order to prove that the Machian solution does not exist in this case. We may restrict our discussions to the early expansion phases, and suppose Eq. (49).

When $\beta>1$ and $a(t)\ll t$, we get from Eq. (58)

$$\Phi(t)=\pm\frac{Mt}{2\pi^2A^3a^2(t)}, \tag{59}$$

and find from Eqs. (17) and (59) after integration

$$\Phi(t)=\Phi_0\exp[\pm 1/(1-\beta)bt^{\beta-1}], \tag{60}$$

which does not obviously satisfy Eq. (56).

When $\beta<1$ and $a(t)\gg t$, from Eq. (58)

$$\Phi(t)=\frac{2M}{\pi^2A^3a(t)},\ or\ 0. \tag{61}$$

For this approximation $(a(t)\gg t)$, we obtain Eqs. (23) and (24) from Eqs. (55) and (56) respectively and find the constraint Eq.(26), which is not compatible with Eq. (61).

When $\beta=1$, Eq. (58) gives

$$\Phi(t)=\frac{M}{\pi^2A^3a(t)}\left(1\pm\sqrt{1+1/4b^2}\right). \tag{62}$$

We can indicate straightforwardly that this equation does not satisfy Eqs. (56) and (55) or (17). Actually, if we put $b=1/\sqrt{3}$, then we obtain

$$\Phi(t) = -\frac{M}{\pi^2 A^3 a(t)} (\sqrt{7}/2 - 1) < 0, \tag{63}$$

which is not consistent with Eq. (47).

Ⅳ Open-Space Case

In similar fashion, the coefficient $A(\omega)$ of the expansion parameter $a(t)$ need to have the following form for the open model ($k=-1$) and $\omega/2+B>0$:

$$\frac{3}{A^2(\omega)} = \frac{\omega}{2} + B. \tag{64}$$

We obtain from Eq. (22) taking this equation into account

$$\frac{\omega}{2}\left[\left(\frac{\dot{\Phi}}{\Phi}\right)^2 + \frac{4\rho}{\Phi} + \frac{1}{a^2}\right] = 3\left(\frac{\dot{a}}{a}\right)^2 + 3\left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{\Phi}}{\Phi}\right) - \frac{3\rho}{\Phi} - \frac{B}{a^2}, \tag{65}$$

which is completely the same equation as in the case of $k=+1$, and $\omega/2+B<0$. Again, therefore, we observe as the solution

$$a(t) = t/\sqrt{3}, \quad B=1, \tag{66}$$

$$\Phi(t)a(t) = -\frac{3M}{2\pi^2 A^3}, \tag{67}$$

and

$$\Phi(t) = -\rho(t)t^2. \tag{68}$$

For $k=-1$, and $\omega/2+B<0$, the coefficient $A(\omega)$ has the form

$$\frac{3}{A^2(\omega)} = -\left(\frac{\omega}{2} + B\right), \tag{69}$$

and we get from Eq. (22)

$$\frac{\omega}{2}\left[\left(\frac{\dot{\Phi}}{\Phi}\right)^2 + \frac{4\rho}{\Phi} - \frac{1}{a^2}\right] = 3\left(\frac{\dot{a}}{a}\right)^2 + 3\left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{\Phi}}{\Phi}\right) - \frac{3\rho}{\Phi} + \frac{B}{a^2}, \tag{70}$$

which is also completely the same as the case of $k=+1$, and $\omega/2+B<0$. We have already known that the Machian solution does not exist in this region.

Finally, we discuss the case that the coefficient $A(\omega)$ of the expansion parameter $a(t)$ does not depend on the coupling parameter ω for the closed and the open spaces ($k=\pm 1$). Let us suppose that the function $a(t)$ absorbs the constant A and then we obtain from Eq. (22)

$$\left(\frac{\dot{\Phi}}{\Phi}\right)^2+\frac{4\rho}{\Phi}\equiv 0, \tag{71}$$

and

$$\left(\frac{\dot{a}}{a}\right)^2+\left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{\Phi}}{\Phi}\right)-\frac{\rho}{\Phi}\equiv -\frac{k}{a^2}\,. \tag{72}$$

From Eqs. (71), (17), and (11), we find after integration in the same way as the flat-space case

$$\Phi(t)\propto t^{-4}, \tag{73}$$

and hence

$$a(t)\propto t^2. \tag{74}$$

Equations (73) and (74) do not satisfy Eq. (72) for $k=\pm 1$. Therefore, we observe that such a Machian solution does not exist in the Brans-Dicke theory.

V Concluding Remarks

Thus, we have proved uniqueness of the Machian solution satisfying $\phi=O(\rho/\omega)$ in the Brans-Dicke theory for arbitrary values of the coupling parameter ω. There are no Machian solutions except {Eqs. (31) and (32)}, which is new, {Eqs. (46) and (48)}, and {Eqs. (66) and (68)} for the flat ($k=0$), closed ($k=+1$), and open ($k=-1$) spaces respectively. The proof is restricted to the case for the homogeneous and isotropic universe with dust matter. It should be noted that the function $\Phi(t)$ does not depend on the coupling parameter ω in the Machian solution.

Each solution has only the lowest order of $\rho/(3+2\omega)$ and is exact for all values of ω.

There is another particular Machian solution for the particular value of the coupling parameter $\omega=-2$, the border between the closed ($\omega<-2$) and the open ($\omega>-2$) spaces, in the case of the flat space. When we extend this solution to a perfect fluid with pressure, its value becomes $\omega=-3/2$ for the radiation universe, which is singular. So we discard this particular solution for the particular value of ω.

We, after all, observe that the Machian solution has always the form $\Phi(t)\propto\rho(t)t^2$. This means that the relation $GM/c^2a=const$ (or $GM/c^2R=const$ for the flat space) is held *for all* t in this cosmological model. First we reached the Machian solution satisfying $\phi=O(\rho/\omega)$ from the constraint $a\phi=const$. Now, inversely, we have necessarily reached the relation $GM/c^2R=const$ (Let us include $GM/c^2a=const.$) from the condition $\phi=O(\rho/\omega)$. They are equivalent to each other, the relation $GM/c^2R=const$ and the postulate that the scalar field exhibits the asymptotic behavior $\phi=O(\rho/\omega)$ for a large value of ω. Let us remember again that this postulate means that a particle in any other empty space does not have the inertial properties (the Machian point of view). On the other hand, the relation $GM/c^2R=const$ is relevant to the global structure of the universe. We find that they are connected closely in the Brans-Dicke theory. The criterion $\phi=O(\rho/\omega)$ would play important roles as a guiding principle to investigate cosmological problems in the Machian point of view.

The scalar field ϕ converges to zero as the universe expands linearly in the closed model. Taking the conservation law $\rho a^3=const$, this is an unavoidable result in the Machian point of view. The universe gradually goes to empty as the universe expands for ever, and particles lose their inertial properties in otherwise empty universe. The behavior of the scalar field $\phi(t) \propto t^{-1}$ means that the gravitational "constant" increases linearly ($G(t) \propto t$) as the universe expands. However, this time-variation of the gravitational "constant" obviously contradicts with the results obtained by recent measurements to detect it [8] ($|\dot{G}/G| \lesssim 1.6 \times 10^{-12} yr^{-1}$).

Moreover, it may be a crucial defect that the coupling parameter ω must be negative ($\omega < -2$) in this closed model. According to particle physics, this means that the scalar field becomes the ghost, the energy-momentum of which is negative. This result is also unavoidable in the Machian point of view. The discussions for the case of $k=+1$ and $\omega < -2$ are parallel to those of $k=-1$ and $\omega > -2$. The two cases split the sign of the gravitational constant. The gravitation becomes an attractive force only for $\omega < -2$ in this Machian solution for the homogeneous and isotropic dust universe in the Brans-Dicke theory.

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