Physical Significance of the Difference between the Brans-Dicke Theory and General Relativity

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Abstract

The asymptotic behavior of the Brans-Dicke scalar field ϕ for the large coupling parameter ω and its physical meaning are discussed for the contracted energy-momentum $T=0$ and $T\neq 0$. The special character of the Brans-Dicke theory, in the Machian point of view, is also discussed in contexts of local and cosmological problems in comparison with general relativity. For cosmological problem, as a reflection of the Machian point of view, we propose a postulate that *the scalar field of a proper cosmological solution should have the asymptotic form* $\phi = O(\rho/\omega)$ *and should converge to zero in the continuous limit* $\rho/\omega \to 0$. It is rather reasonable and essential that the BD does not reduce to GR for large ω. For local problems, as a reflection of the presence of cosmological matter in the universe, we require another postulate that *the scalar field by locally-distributed matter should show the asymptotic behavior* $\phi = \langle\phi\rangle + O(1/\omega)$. Then, we can discuss local problems with an asymptotically-flat ($r\to\infty$) boundary condition for the metric $g_{\mu\nu}$ and $\langle\phi\rangle \approx G^{-1}$ for the scalar field, without considering the cosmological background environment. For local problems, BD reduces properly to GR for large ω.

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I Introduction

It seems that Einstein's general relativity has increasingly obtained its exactness and validity by many experimental and observational tests. Nevertheless, on the other hand, much efforts for scalar-tensor theories of gravination also continue for a long time. We have some historical or fundamental bases on which we believe that there should exist some kinds of scalar field as the gravitational field.

The Brans-Dicke theory [1] is the prototype of such scalar-tensor theories of gravitation, and the gravitational field is described by the metric tensor $g_{\mu\nu}$ of the Riemannian manifold and the non-minimally coupled scalar field ϕ on that manifold, which represents the spacetime- varying gravitational "constant". The field equations of the Brans-Dicke theory are obtained by the similar variational method as the Einstein theory, and given as following in our sign conventions:

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=\frac{8\pi}{c^4\phi}T_{\mu\nu}-\frac{\omega}{\phi^2}\left(\phi_{,\mu}\phi_{,\nu}-\frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}\right)$$
$$-\frac{1}{\phi}(\phi_{,\mu;\nu}-g_{\mu\nu}\Box\phi), \tag{1}$$

$$\Box\phi=-\frac{8\pi}{(3+2\omega)c^4}T, \tag{2}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of matter and ω is the coupling parameter of the scalar field.

As mentioned in many literatures (see, for example, [2]), when the coupling parameter ω is large enough, the scalar field and the field equation of gravitarion have the following approximate form:

$$\phi=\langle\phi\rangle+O(1/\omega), \tag{3}$$

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=\frac{8\pi}{c^4\phi}T_{\mu\nu}+O(1/\omega), \qquad (4)$$

and in the limit of infinity ($\omega\rightarrow\infty$), the scalar field ϕ converges to constant $\langle\phi\rangle$, thus the field equations of gravitation coincide completely with those of general relativity by replacing ϕ with Newton's gravitational constant $G\equiv\langle\phi\rangle^{-1}$.

Recently, however, some authors [3], [4], reported that these discussions are generally not right when the contracted energy-momentum tensor $T=T_\mu^\mu$ vanishes. According to Banerjee and Sen [3], in this situation $T=0$, asymptotic behavior of the scalar field becomes

$$\phi=\langle\phi\rangle+O(1/\sqrt{\omega}) \qquad (5)$$

when the coupling parameter is large enough. In the limit of infinity, though the scalar field definitely converges to constant, the second term of the right-hand side of Eq. (1) remains nonvanishing and the field equations of the Brans-Dicke theory do not coincide with those of the Einstein theory with the same energy-momentum tensor $T_{\mu\nu}$. As for such examples of exact solutions, see Refs of [3], [4]. They say that *the condition* $T\neq 0$ *is both neccessary and sufficient for the Brans-Dicke solutions to yield the corresponding solutions of general relatibity with the same* $T_{\mu\nu}$ *in the infinite* ω *limit.*

However, this theorem is not true as indicated by Faraoni [4] with a counterexample [5]. Faraoni gave a rigorous mathematical proof to the asymptotic behavior Eq. (5) by discussing the conformal invariance of the Brans-Dicke theory when $T=0$. He insists only that *the Brans-Dicke solutions with* $T=0$ *generically fail to the corresponding solutions of general relativity when* $\omega\rightarrow\infty$.

In this paper, we survey literatures and discuss generally the physical meaning of the relationship between the Brans-Dicke theory and the Einstein theory in the cases $T=0$ and $T\neq 0$ in contexts of local or cosmological problems, and then make clear the essence of the Brans-Dicke theory. We indicate another cosmological counterexample to the above theorem [3] in section IV. However its physical meaning is completely different from those of the above authors, who belong to the second standpoint in section V. We insist that the cosmological solutions in the Brans-Dicke theory which do not reduce to the corresponding solutions in general relativity when $\omega\to\infty$ are rather reasonable and essential. Going back to the original motivation of the Barns- Dicke theory, the Machian point of view, we realize that a particle in otherwise empty space should not have the inertial properties. We will propose two postulates for local and cosmological problems respectively.

II Physical Meaning of the Asymptotic Behavior

Let us discuss the asymptotic behavior of the scalar field when $T=0$. An order of magnitude estimate by Banerjee and Sen [3] is more appropriate to understand its physical meaning. When $T=0$, we obtain from Eq. (2)

$$\Box\phi=0, \tag{6}$$

and get from the trace of Eq. (1)

$$R=-\frac{8\pi}{c^4\phi}T-\frac{\omega}{\phi^2}\phi_{,\lambda}\phi^{,\lambda}-\frac{3}{\phi}\Box\phi. \tag{7}$$

It is easy to see asymptotic behavior Eq. (5) of the scalar field from this equation when $T=0$. However, remember we assume tacitly that the scalar field ϕ converges to constant in the infinite ω limit and the scalar curvature R does not depend on ω, both of which do not seem to be obvious. Moreover, the Minkowski space with $T=0$ and $R=0$ has only the constant scalar $\langle\phi\rangle$, which is independent of the coupling parameter ω. It is to be remarked that a solution satisfying

$$\frac{\omega}{\phi^2}\left(\phi_{,\mu}\phi_{,\nu}-\frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}\right)=0 \quad (8)$$

is only $\phi=const$ [4]. Therefore, all Einstein spaces ($R_{\mu\nu}=0$) with constant ϕ are exceptions for the statement by Banerjee and Sen, or by Faraoni.

When $T\neq 0$ the asymptotic form of the field equation (2) becomes

$$\Box\phi=O(1/\omega), \quad (9)$$

and we observe the well-known asymptotic behavior Eq. (3) of the scalar field. We, however, should strictly read Eq. (9) as $\Box\phi=O(T/\omega)$, or for simplicity

$$\Box\phi=O(\rho/\omega) \quad (10)$$

for dust matter.

Now we can understand the physical meaning of the difference of the asymptotic behavior of the scalar field. The Brans-Dicke theory includes originally the coupling parameter ω in the right-hand side of Eq. (2) and in the second term of the right-hand side of Eq. (1). When $T\neq 0$, the dependence of ω in the scalar field comes fundamentally from the coupling parameter ω in the right-hand side of Eq. (2), and the second team of the right-hand side of Eq. (1) vanishes in the infinite ω limit. When $T=0$, the right-hand side of Eq. (2) vanishes and the dependence of ω comes fundamentally from ω in the second term of the right-hand side of Eq. (1). The Brans-Dicke scalar field has finite indefiniteness $\phi_V(x^u)$ which satisfies the d'Alembertian equation (6) even when matter does not exist ($T=0$). This scalar field $\phi_V(x^u)$ is constrained by another field equation (1), and thus it has the dependence of ω like Eq. (5). The scalar field $\phi_V(x^u)$, which behaves like a source of the gravitational field $g_{\mu\nu}$ in Eq. (1), has no material origin. The constant part $\langle\phi\rangle$ itself is a special case of this scalar field without material origin. To the contrary, the asymptotic behavior of the scalar field with material origin is determined by its field equation (2) with the source term.

In general, when matter exists ($T \neq 0$), the scalar field includes a part given by matter and indefiniteness $\phi_V(x^u)$ for $T=0$, and its asymptotic behavior for large ω becomes

$$\phi = \langle\phi\rangle + O(1/\sqrt{\omega}) + O(1/\omega). \qquad (11)$$

It is clear that the term of $O(1/\sqrt{\omega})$ is more dominant than that of $O(1/\omega)$ when the coupling parameter ω is large enough. Therefore, even if $T \neq 0$, the Brans-Dicke theory fails to yield general relativity in the infinite ω limit. This situation could produce other possible counterexamples, $T \neq 0$ and $\phi = \langle\phi\rangle + O(1/\sqrt{\omega})$, to the theorem proposed by Banerjee and Sen. An example cited by them to reinforce the theorem, a closed vacuum ($T=0$) Friedmann-Robertson-Walker solution with cosmological constant Λ [6], [7], should rather be included because of $\Box\phi = 2\Lambda\phi/(2\omega+3) \neq 0$ though $T=0$.

III Local Problems

When we consider the difference between the Brans-Dicke theory and the Einstein theory, we had better distinguish between local problems (with locally-distributed matter, like black holes) and cosmological (or global) problems. Brans and Dicke [1] also comment, in discussing a Schwarzschild solution in their theory, that we premise the existence of distant matter in the universe. Our universe always exists, and in the Brans-Dicke theory we need discuss local problems in the presence of cosmologically-distributed matter, which supports the gravitational "constant". In the framework of general relativity, which has *a priori* gravitational constant, we need not consider our environment of the universe and discuss purely the local gravitational field with an asymptotically-flat boundary condition.

It might be very difficult to solve globally all configurations of matter in the universe in the Brans-Dicke theory. However, it seems to be a good enough approximation to divide the two side, local and cosmological problem, because our universe is huge enough. Locally-distributed matter does not almost influence the structure of the whole universe. We can consider independently the structure of the whole universe and determine a cosmological model in the Brans-Dicke theory. After that (or the other hand), we can discuss individual problems of locally-distributed matter with an asymptotically-flat boundary condition without considering the environment of the universe. To do so, we need accept two premises; We use an experimental value of gravitational constant supported by cosmological matter for the constant scalar field $\langle\phi\rangle = 1/G$, and require that local scalar field ϕ also converges to $\langle\phi\rangle$ at the distant enough region ($r \to \infty$). Moreover, we need adopt a selection rule that the scalar field should show the asymptotic behavior $\phi = \langle\phi\rangle + O(1/\omega)$, which is also a reflection of the presence of cosmological matter in the universe.

Let us consider the static spherically symmetric vacuum solution [1] in the Brans-Dicke theory (only scalar part):

$$\phi = \phi_0 \left(\frac{1 - B/r}{1 + B/r} \right)^{-C/\sigma} \tag{12}$$

where $B = (M/2C^2\phi_0)\,[(2\omega+4)/(2\omega+3)]^{1/2}$, $\sigma = [(C+1)^2 + C\,(1 - \frac{1}{2}\omega C)]^{1/2}$, and C is arbitrary constant. It is obvious that this solution converges to the constant ϕ_0 in the infinite ω limit. In a case of arbitrary constant C (independent of ω), the asymptotic form of this solution becomes $\phi=\langle\phi\rangle+O(1/\sqrt{\omega})$ [3], which means that Eq. (12) does not produce the corresponding solution of general relativity, the Schwarzschild solution. We should suppress this behavior and indefiniteness of solutions with no material origin. We can not accomplish this work by introducing additional boundary conditions. Our selection rule acts to choose a proper solution (or solutions) here. We should select only a solution the scalar field of which shows the asymptotic behavior $\phi=\langle\phi\rangle+O(1/\omega)$ with material origin, and which yields the corresponding solution of general relativity in the infinite ω limit. Because the corresponding exact and global Brans-Dicke solution is originally generated by the nonvanishing energy-momentum tensor ($T_{\mu\nu}\neq 0$), with locally-distributed matter and cosmological matter in the universe, and so should have the asymptotic form $\phi=\langle\phi\rangle+O(1/\omega)$. Even the Schwarzschild-like solution is originally not a pure local problem in the Brans-Dicke theory. However, we can forget the effect of the presence of cosmological matter if we set the two postulates for local problems in the Brans-Dicke theory. In this standpoint, general relativity is the self-complete approximate-theory of gravitation as it needs no additional postulates.

If we take formally a choice $C=-1/(2+\omega)$, the equation (12) behaves asymptotically as Eq. (3) for the large enough ω and the whole solution becomes identical with the Schwarzschild solution of the Einstein theory for $\omega\to\infty$ [1], [6]. Another formal choice $C=-1/2\omega$ is also available [4]. However, it is to be remarked that this ω has no meanings though the same letter ω is used. This is not the actual coupling parameter ω derived from the right-hand side of Eq. (2) or the second term of the right-hand side of Eq. (1) as long as we consider exactly a point-mass M in the empty space. However, if we regard Eq. (12) as the local solution *approximated* for *a point-mass M at the origin* and *cosmological matter in the universe*, we may be able to interpret this ω as the real coupling parameter derived from the right-hand side of Eq. (2) with source matter. As it is too difficult to solve exactly all configurations of matter in the universe in the Brans-Dicke theory, this is nothing but a conjecture, but let us adopt as a postulate. The coupling parameter ω for approximated local problems is actual as a reflection of the presence of cosmological matter in the universe and the scalar field shows the asymptotic behavior $\phi=\langle\phi\rangle+O(1/\omega)$. In this meaning, we should discuss local problems "with $T=0$" and we can restrict indefiniteness with no material origin. There exists arbitrariness of forgiven solutions, for example a choice of C, and it remains conrtoversial. They give different solutions for finite values of ω, though the corresponding solution of general relativity in the infinite ω limit is same.

For local problems, they are equivalent to each other for a proper solution to have the material origin $T\neq 0$ (in the above meaning), to behave asymptotically as $\phi=\langle\phi\rangle+O(1/\omega)$ for the large enough ω, and to converge to the corresponding solution of general relativity in the infinite ω limit. The contracted energy-momentum $T=0$ (in the exact meaning) is not identical with the asymptotic behavior $\phi=\langle\phi\rangle+O(1/\sqrt{\omega})$, which leads necessarily to the fact that the Brans-Dicke solutions fail to reduce to the corresponding solutions of general relativity when $\omega\to\infty$.

Banerjee and Sen, or Faraoni seem to apply the Brans-Dicke theory to pure local problems for locally-distributed matter without any restriction. They treat exactly local problems as whole cosmological problems. They discuss exact solutions with exact $T=0$. In their common standpoint, the statement $T=0$ means that there exist completely no other matter in the universe. To the contrary, though we use the statement $T=0$, this is a local approximation for local problems and actually the complete global contracted energy-momentum tensor does not vanish ($T\neq 0$) because of the presence of other cosmological matter in the universe. When we discuss exactly locally-distributed matter in otherwise empty space, the scalar field ϕ should not have the constant scalar field $\langle\phi\rangle$. This is the important keynote to understand the Brans-Dicke theory true. It is meaningless that we consider strictly the situation in which matter does not exist, or vacuum space in the Brans-Dicke theory. These situations become essential for cosmological problems

IV Cosmological Problems

Next we consider cosmological problems to make clear further the essence of the Brans-Dicke theory. Let us discuss first the Brans-Dicke flat solution [1] for the homogeneous and isotropic universe. Assuming the initial conditions

$$\phi=a=0\ ;\ t=0, \tag{13}$$

it is given as

$$ds^2=-dt^2+a^2(\mathrm{t})[d\chi^2+\chi^2(d\theta^2+\sin^2\theta d\varphi^2)], \tag{14}$$

$$\phi=\phi_0(t/t_0)^r,\ a=a_0(t/t_0)^q,\ \rho a^3=\rho_0 a_0{}^3, \tag{15}$$

with

$$\mathrm{r}=2/(4+3\omega),\ q=(2+2\omega)/(4+3\omega), \tag{16}$$

and

$$\phi_0 = 4\pi[(4+3\omega)/(3+2\omega)c^2]\rho_0 t_0^2, \tag{17}$$

when ρ_0 is the present mass desity. For the large coupling parameter ω, it is easy to observe that the scalar field ϕ of solution behaves like Eq. (3) [6]. If the mass density ρ_0 decreases to zero, the scalar field ϕ also conveges to zero, and this situation is suitable for the material origin of the scalar field. It is well-known that this solution reduces to the Einstein-de Sitter universe of general relativity in the infinite ω limit. However, what does the constant value $\langle\phi\rangle = 6\pi\rho_0 t_0^2/c^2$ mean? In the infinite ω limit, the coupling betwen the scalar field and matter vanishes. Why does the mass density ρ_0 appear in the constant scalar field $\langle\phi\rangle$? Which value of the density should we take? This situation is rather strange as to the material origin. After all, this constant scalar field $\langle\phi\rangle$ seems to be merely constant which has no mterial origin.

O'Hanlon and Tupper [8] solution for a vacuum, spatially flat Friedmann-Robertson-Walker spacetime has the asymptotic behavior $\phi = \langle\phi\rangle + O(1/\sqrt{\omega})$ [6], which means that this solution has no material origin. This solution also has the constant scalar field for $\omega \to \infty$. Nariai [9] flat solution with a perfect fluid has the asymptotic behavior Eq. (5) for $T=0$ (radiation), and Eq. (3) for $T \neq 0$ (matter) [3]. In both cases the scalar field converges to the constant ϕ_0 in the infinite ω limit.

The following cosmological solution [10], [11] gives a counterexample to the theorem by Banerjee and Sen [3] and moreover, an interesting example as to the material origin of inertia. Dehnen and Obregón say that this model has no analogy in general relativity [10], but this is not adequate [12]. The Brans-Dicke theory has a particular closed solution for the homogeneous and isotropic universe with dust ($T=\rho c^2$), satisfyung $a(t)\phi(t) = const$,

$$ds^2=-dt^2+a^2(t)[d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta d\varphi^2)], \tag{18}$$

$$\phi(t)=-[8\pi/(3+2\omega)c^2]\rho(t)t^2, \tag{19}$$

$$a(t)=-[2/(2+\omega)]^{1/2}t, \tag{20}$$

$$2\pi^2a^3(t)\rho(t)=M, \tag{21}$$

with

$$\omega<-2,\ \ G(t)M/c^2a(t)=\pi, \tag{22}$$

where the gravitational "constant" $G=(4+2\omega)/(3+2\omega)\phi$ and M is the total mass of the universe. The scalar field has obviously the asymtotic behavior $O(1/\omega)$, but does not have the constant value $\langle\phi\rangle$ in the infinite ω limit ($\omega\rightarrow-\infty$). The expansion parameter a also has the ω -dependence, which means the scalar curvature R itself has the ω-dependence.

Let us write down the nonvanishing components of the field equations (1) and (2) for the metric Eq. (18) to discuss the details of the asymptotic behavior:

$$2a\ddot{a}+\dot{a}^2+1=-\frac{8\pi}{(3+2\omega)c^2}\frac{a^2\rho}{\phi}-\frac{1}{2}\omega a^2\left(\frac{\dot{\phi}}{\phi}\right)^2+a\dot{a}\left(\frac{\dot{\phi}}{\phi}\right), \tag{23}$$

$$\frac{3}{a^2}(\dot{a}^2+1)=\frac{16\pi(1+\omega)}{(3+2\omega)c^2}\frac{\rho}{\phi}+\frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2+\frac{\ddot{\phi}}{\phi}, \tag{24}$$

$$\ddot{\phi}+3\frac{\dot{a}}{a}\dot{\phi}=\frac{8\pi}{(3+2\omega)c^2}\rho, \tag{25}$$

where a dot donotes the derivative with respect to t. For small enough coupling parameter ($\omega\ll-1$), we can estimate the order of each terms by means of the solution, for example,

$$\frac{8\pi}{(3+2\omega)c^2}\frac{a^2\rho}{\phi}\sim O(1/\omega), \tag{26}$$

$$\frac{1}{2}\omega a^2\left(\frac{\dot{\phi}}{\phi}\right)^2 \sim O(1), \tag{27}$$

$$a\dot{a}\left(\frac{\dot{\phi}}{\phi}\right) \sim O(1/\omega), \tag{28}$$

$$a^2\frac{\ddot{\phi}}{\phi} \sim O(1/\omega). \tag{29}$$

Remark that the mass density ρ is given and does not depend on ω, and so M has the ω-dependence derived from a. The term Eq. (27), which is the contribution from the second term of the right-hand side of Eq. (1), remains nonvanishing even though the scalar field has the asymptotic behavior $O(1/\omega)$.

If we put $\lambda(t) \equiv -(\omega/2)(\dot{\phi}/\phi)^2$ and $\kappa(t) \equiv 8\pi/c^4\phi$, we get from Eq. (23) and Eq. (24)

$$2a\ddot{a} + \dot{a}^2 - \lambda a^2 + 1 = O(1/\omega), \tag{30}$$

$$\frac{3}{a^2}(\dot{a}^2 + 1) + \lambda = \kappa\rho c^2 + O(1). \tag{31}$$

Thus we obtain in infinite ω limit ($\omega \to -\infty$)

$$\lambda a^2 = 1, \quad \kappa\rho c^2 a^2 = 4. \tag{32}$$

If we regard λ as the cosmological "constant", these relations are similar to those of the static Einstein universe with negligigble pressure in general relativity except the difference of the redius of the universe in $\sqrt{2}$ which is derived from the opposite sign of λ in Eq. (31). In the infinite ω limit, the expansion parameter a reduces to zero, but this is due to the initial condition $a=0$ at $t=0$ [12]. There exists a discrete and isolated limit at $\omega \to -\infty$. For the finite ω ($\omega < -2$) we observe

$$\lambda(t)\,a^2(t) = \omega/(2+\omega), \quad \kappa(t)\,\rho(t)\,c^2 a^2(t) = 4, \tag{33}$$

and so the effective cosmological "constant" $\lambda(t)$ decreases rapidly as the universe expands.

It is remarkable that the scalar field ϕ of this solution converges to zero for both cases in which the mass density ρ goes to zero, and in which the coupling parameter ω goes continuously to the infinity ($\omega \to -\infty$). This means that the combination of ρ/ω plays an important role there. This situation is rather preferable for the material origin of the scalar field. We insist, in the Machian point of view, that a proper cosmological solution should have the asymptotic from $\phi = O(\rho/\omega)$ without the constant part $\langle\phi\rangle$. This postulate is a direct reflection of the statement that a particle in otherwise empty space does not have inertia. The Brans-Dicke theory becomes singular when the scalar field ϕ converges to zero as the ratio ρ/ω vanishes. This is the very necessary situation that we expect in relation to the material origin of inertia.

Now we need consider the correspondence between the Brans-Dicke theory and general relativity in combinations of $\langle\phi\rangle \neq 0$ or $\langle\phi\rangle = 0$, and $O(1/\omega)$ or $O(1/\sqrt{\omega})$. Let us put for abbreviation

$$A_{\mu\nu} \equiv \frac{\omega}{\phi^2}\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2} g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}\right), \tag{34}$$

$$B_{\mu\nu} \equiv \frac{1}{\phi}(\phi_{,\mu;\nu} - g_{\mu\nu}\Box\phi), \tag{35}$$

$$C \equiv \frac{8\pi}{c^4\phi}. \tag{36}$$

We can summarize orders of magnitude of each terms in ω or $\sqrt{|\omega|}$ as following:

case(i) $\phi = \langle\phi\rangle + O(1/\omega)$, $\langle\phi\rangle \neq 0$,

$$A_{\mu\nu} \sim O(1/\omega),\ B_{\mu\nu} \sim O(1/\omega),\ C \sim O(1), \tag{37}$$

case(ii) $\phi = \langle\phi\rangle + O(1/\sqrt{|\omega|})$, $\langle\phi\rangle \neq 0$,

$$A_{\mu\nu} \sim O(1),\ B_{\mu\nu} \sim O(1/\sqrt{|\omega|}),\ C \sim O(1), \tag{38}$$

case(iii) $\phi = O(1/\omega)$,

$$A_{\mu\nu} \sim O(\omega),\ B_{\mu\nu} \sim O(1),\ C \sim O(\omega), \tag{39}$$

case(iv) $\phi = O(1/\sqrt{|\omega|})$,

$$A_{\mu\nu} \sim O(\omega),\ B_{\mu\nu} \sim O(1),\ C \sim O(\sqrt{|\omega|}). \tag{40}$$

These results are derived on the assumption that the metric tensor $g_{\mu\nu}$ converges to a nonvanishing function in the infinite ω limit. If $g_{\mu\nu}$ has other ω-deoendence which does not satisfy this assumption, we need another individual analysis for the specific solution, and the result seems to become different from the above. Even for local problems we can not deny this possibility. However, it is likely that $g_{\mu\nu}$ converges to a nonvanishing function in the infinite ω limit if the scalar field ϕ converges to $\langle\phi\rangle \neq 0$ for local problems. Anyhow, it is common that the Brans-Dicke solutions fail to reduce to the corresponding solutions of general relativity when $|\omega| \to \infty$.

V Discussions

Should the Brans-Decke theory reduce to general relativity in the infinite ω limit ? No longer, its statement seems to be a preconception. It is true that general relativity goes to the Newtonian theory of gravitation in the weak field approximation ($GM/Rc^2 \ll 1$). The fact that both general relativity and the Newtonian theory have the common parameter, Newton's gravitational constant G, makes it possible. However, the Brans-Dicke theory and general relativity do not have a common parameter each other. The infinite limit of the coupling parameter ω is ambiguous. After all, it is natural to realize that the Brans-Dicke theory is a different theory of gravitation from general relativity and need not necessarily reduce to it in the infinite ω limit.

Whether *the Brans-Dicke solutions reduce to the corresponding* *solutions of general relativity in the infinite ω limit* is closely connected to their *material origin.* For both local and cosmological problems, the asymptotic behavior $\phi = \langle\phi\rangle + O(1/\sqrt{\omega})$ for the large coupling parameter ω of the scalar field, which leads to the difference in the infinite ω limit, is derived from a part with no material origin, even if $T \neq 0$. For cosmological problems, the asymptotic form $\phi = O(\rho/\omega)$ without the constant part $\langle\phi\rangle$, which does not include *a priori* material origin and is completely determined by cosmological matter itself in the universe, leads to the difference in the infinite ω limit, when a connection with matter is cut off. The Brans-Dicke theory can manipulate principally the material origin itself of the field, and on the other hand, general relativity has the given gravitational constant G which has the *tacit* materal origin (the third standpoint, later). The differences of two theories are rather essential in the physical meaning, the material origin, for cosmological problems. However, taking the experimental and observational data into account, it is preferable for local problems that the Brans-Dicke theory reduces to general relativity in the infinite ω limit.

There exist at least three standpoints; First, standard general relativity is the complete classical theory of gravitation. Second, the Brans-Dicke theory is complete by itself. Third, the Brans-Dicke theory with some additional conditons produces physically reasonable solutions.

The first standpoint is the simplest and the most real, even though we can not understand the origin of the gravitational constant G and cannot help accepting its value *a priori*. The scalar field does not exist as the gravitational field. We do not need the redundant Brans-Dicke theory. General relativity has no restritions on applicable range and is completely valid not only for local problems but also for cosmological problems. Even a particle in the Minkowski space has inertia. However, there exist no contradictions in general relativity. The present experimental and observational tests strongly support this standpoint.

In the second standpoint, the opposite extreme to the first, we can apply the Brans-Dicke theory to all kinds of problems with no restrictions and formally obtain their solutions if possible. We may set *exactly* asymptotically-flatness as a boundary condition for locally-distributed matter. We may investigate black holes in otherwise empty space. We can discuss even a vacuum space itself. It may be the Minkowski space. May a particle show the inertial property in this space? These situations are similar to general relativity. However, a crucial difference exists between this standpont of the Brans-Dieke theory and the first standpoint of general relativity. In the Brans-Dicke theory, we encounter a serious difficultly owing to ambiguity with $\Box\phi = 0$. This ambiguity on solutions is not avoidable as long as we consider vacuum ($T = 0$), even if we suppose specific boundary conditions for the scalar field. This is a crucial defect of the Brans-Dicke theory in the second standpoint. The Brans-Dicke theory is not complete by itself. In this ambiguity with $T = 0$, the scalar field behaves like $\phi = \langle\phi\rangle + O(1/\sqrt{\omega})$ for the large coupling parameter ω. This form of the scalar field leads to a difference with the corresponding solution of general relativity with the same energy-momentum tensor in the infinite ω limit. We might be able to check this difference by measurements.

The third standpoint set the presence of matter in the universe forth as a premise; This is the Machian point of view. The presence of matter in the universe gives a particle inertia. If the mass density of matter decreases, inertia of a particle should also decrease. A particle should not have inertia in otherwise empty space. Cosmology has a special situation in physics. We always live in this universe and can not test alternatives. We cannot help discussing every physical phenomena in this environment. We cannot consider pure empty space and it has no meanings. We realize that the Brans-Dicke theory itself does not automatically satisfy these situations. We need to require some additional conditions to complete the theory. We had better clearly distinguish between local and cosmological problems. We propose two postulates for local and cosmological problems respectively.

For local problems, we restrict to solutions whose *the scalar field shows the asymptotic behavior* $\phi=\langle\phi\rangle+O(1/\omega)$. This selection rule is a reflection of the presence of cosmological matter in the universe and suppresses ambiguity with $\Box\phi=0$. Moreover, we presume that *the cosmological background gives the finite reasonable scalar field* $\langle\phi\rangle$, that is, the gravitational constant G at the present time. Thus we can handle individual local problems with asymptotically-flat boundary conditions for the metric tensor $g_{\mu\nu}$ and the boundary condition for the scalar field ϕ ($\phi\rightarrow\langle\phi\rangle$ as $r\rightarrow\infty$) in the Brans-Dicke theory without considering our environment. This split between local problems and the cosmological background is an extremely good approximation in our universe. We may discuss the space-varying G by locally-distributed matter in the "empty" space (in this premise).

For cosmological problems, we require the postulate that *the scalar field of a proper cosmologacal solution should have the asymptotic form* $\phi=O(\rho/\omega)$. This is a reflection of the Machian point of view. If the inertial properties are determined completely by the presence of matter in the universe, the scalar field should not include the constant $\langle\phi\rangle$

which has no material origin (This is a cosmologcal problem.). The scalar field ϕ should converge to zero when the mass density ρ decreases to zero in a proper cosmological model. The scalar field should also converge to zero when the coupling parameter $|\omega|$ diverges to infinity and the connection between the scalar field and matter vanishes. The mass density ρ and the coupling parameter ω are closely connected as the source term of Eq. (2) and so we can combine two conditions. Thus, we require that *the cosmological scalar field should converge to zero in the continuous limit* $\rho/\omega \to 0$. This requirement is crucial for considering the difference between general relativity and the Brans-Dicke theory in the infinte limit of ω. The Minkowski space which has the constant scalar field is excluded as a proper solution. The cosmological solutions which reduce to the corresponding solutions of general relativity in the infinite limit of ω are also excluded. A cosmological solution which fails to the corresponding solution of general relativity is rather physically reasonable and essential The constant scalar $\langle\phi\rangle$ may be derived from the contribution of quantum corrections. However, this contribution should be classically renormalized to the mass density because the inertial-frame dragging is dominated completely by the dstribution of the mass density itself [13].

The role of general relativity is clear in the third standpoint. All proper solutions satisfying $\phi=\langle\phi\rangle+O(1/\omega)$ for *local problems* reduce to the corresponding solutions of general relativity in the infinitie ω limit (, fixing cosmological part), so general relativity is the complete approximate-theory of gravitation in this standpoint (GR needs no additional conditions). Rather, this fact may be more fundamental as a postulate for local problems. General relativity is effective enough for local problems and for a small period for which the universe is quasi- static and the gravitational "constant" is constant enough.

It is a matter of course that experimental and observational tests should finally determine which theory (, including other extended scalar-tensor theories) is true and which standpoint is appropriate. After all, the essential difference may appear only in cosmological problems owign to the experimentally established large coupling parameter ω. We have not known an exact solution for our universe yet.

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