The quantities W, L, K and variations of geodesics in Finsler spaces

Tetsuya NAGANO

Abstract

The author investigates the first and the second variations of the arc length of curves under the standpoint of linear parallel displacements. Last year the author studied linear parallel displacements along an infinitesimal parallelogram and obtained three quantities on $\mathcal{H}$([9],[10]). In this paper, they appear in the second variation.

Keywords and phrases : linear parallel displacement, first variation, second variation, locally Minkowski space, Finsler geometry.

Introduction

The author have been studying the linear parallel displacement in Finsler geometry from 2008. In [4] the author called it "parallel displacement". But in [6] and [7] the author renamed it "linear parallel displacement" because Prof.Z.Shen had already given the nearly same definition satisfying the linearity by the coefficients N^i_j in his book [1] . He called such parallel displacements "linear parallel".

First, we put terminology and notations used in this paper(cf.[2] and [3]). Let M be an n-dimensional differentiable manifold and $x = (x^i)$ a local coordinate of M. TM is the tangent bundle of M and $(x, y) = (x^i, y^i)$ is a local coordinate of TM. $N = (N^i_j(x, y))$ is an non-linear connection of TM and its coefficients of N on a local coordinate (x, y). $F(x, y)$ is a Finsler structure (or Finsler metric, Finsler fundamental function) on M. Further, $F\Gamma = (N^i_j(x, y), F^i_{jr}(x, y), C^i_{jr}(x, y))$ is Finsler connection and its coefficients of $F\Gamma$ satisfying $T^i_{rj} := F^i_{rj} - F^r_{jr} = 0$, $D^i_j := y^r F^i_{rj} - N^i_j = 0$ and $g_{ij|k}(x, y) = 0$(h-metrical). And $N^i_j(x, y), F^i_{jk}(x, y), C^i_{rj}(x, y)$ are positively homogeneous of degree 1, 0 and –1, respectively. Therefore N^i_j and F^i_{jr} come to Cartan's ones. Last, we denote the collection of horizontal vectors at every point on TM by $\mathcal{H}$. This is the subbundle of TTM and its dimension is $3n$. So we denote a local coordinate of $\mathcal{H}$ by (x, y, z). And it is called "horizontal subbundle of TTM". All of objects appeared in this paper (curves, vector fields, etc) are differentiable. In additions, indexes $a, b, c, \cdots, h, i, j, k, l, m, \cdots, \alpha, \beta, \cdots$, run on from 1 to $n = \dim M$.

Now, for a vector field on a curve c with a parameter t, we give a following definition of linear parallel displacements along c([4],[5],[6],[7]).

Definition 0.1 *For a curve $c = (c^i(t))$ $(a \leq t \leq b)$ on M and a vector field $v = (v^i(t))$ along c, if the equation*

$$\frac{dv^i}{dt} + v^j F^i_{jr}(c, \dot{c})\dot{c}^r = 0 \qquad (\dot{c}^r = \frac{dc^r}{dt}) \tag{0.1}$$

is satisfied, then v is called a parallel vector field along c, and we call the linear map $\Pi_c : v(a) \longrightarrow v(b)$ a linear parallel displacement along c.

The difference from the traditional notion of parallel displacement in Finsler geometry are three points. One of them is that the inverse vector field $v^{-1}(\tau)(\tau = -t + \alpha)$ is not always parallel along the inverse curve $c^{-1}(\tau)$, even if $v(t)$ is parallel vector field along a curve $c(t)$(cf.[4], [6]).

The others of them are that we can consider, for vector fields $u(t),\ v(t)$ along $c(t)$, an inner product $g_{ij}(c, \dot{c})u^i(t)v^j(t)$ along the curve $c(t)$ and the inner product is not always preserved, even if u, v are parallel vector fields along a curve c.

Then we have(cf.[5])

Proposition 0.1 *Let $(M, F(x, y))$ be a Finsler space with a Finsler connection $(N^i_j, F^i_{jr}, C^i_{jr})$ satisfying h-metrical $g_{ij|r} = 0$. For any parallel vector fields $v = (v^i(t)),\ u = (u^i(t))$ along a curve $c = (c^i(t))$, if c is a path or a geodesic, then the inner product $g_{ij}(c, \dot{c})v^iu^j$ along c is preserved.*

Theorem 0.1 *Let $(M, F(x, y))$ be a Finsler space with a Finsler connection. We assume that the Finsler connection is h-metrical and the metric g_{ij} is positive definite. Any smooth curve preserves the inner products of parallel vector fields along it, if and only if, $\frac{\partial g_{ij}}{\partial y^r} = 0$ is satisfied, namely, (M, g_{ij}) is a Riemannian space.*

1 Linear parallel displacements along an infinitesimal parallelogram

Next we introduce conclusions obtained by investigating linear parallel displacements along an infinitesimal parallelogram([9],[10]).

We studied two cases. One is the case that makes an initial vector be two parallel vector fields along two routes(Case I), and the other is the case making a parallel vector field along one loop(Case II). Hereafter, we assume all points and curves are in one coordinate neighborhood.

First, we define three quantities W, L, K on $\mathcal{H}$ as follows:

$$W^i_{hj}(x, y, z) := F^i_{hj}(x, y) - F^i_{hj}(x, z), \tag{1.1}$$

$$L^i_{hj}(x, y, z) := \frac{\partial F^i_{hm}}{\partial y^j}(x, y)z^m + \frac{\partial F^i_{hm}}{\partial z^j}(x, z)y^m, \tag{1.2}$$

$$K^i_{hjk}(x, y, z) := \frac{\delta F^i_{hj}}{\delta x^k}(x, y) - \frac{\delta F^i_{hk}}{\delta x^j}(x, z) - F^i_{mj}(x, y)F^m_{hk}(x, y) + F^i_{mk}(x, z)F^m_{hj}(x, z). \tag{1.3}$$

Case I. Let p, q, r, s be four points on M and let $(x^i), (x^i + \xi^i), (x^i + \xi^i + \eta^i), (x^i + \eta^i)$ be their coordinates, respectively. Further c_1, c_2, c_3, and c_4 are following curves with a parameter t $(0 \leq t \leq 1)$:

$$(I) \begin{cases} c_1(t) : & x^i(t) = x^i + t\xi^i \ (p \text{ to } q), \\ c_2(t) : & x^i(t) = x^i + \xi^i + t\eta^i \ (q \text{ to } r), \\ c_3(t) : & x^i(t) = x^i + t\eta^i \ (p \text{ to } s), \\ c_4(t) : & x^i(t) = x^i + \eta^i + t\xi^i \ (s \text{ to } r). \end{cases}$$

We take two routes $c = c_1 + c_2(p \to q \to r)$ and $\bar{c} = c_3 + c_4(p \to s \to r)$, and consider linear parallel displacements along c and $\bar{c}$, respectively. Let $V = (V^i)$ be an initial vector at p and let V_q, V_r be the values at q and r by the parallel vector field along c, respectively. Further, let $\bar{V}_s, \bar{V}_r$ the value at s and r by the parallel vector field along $\bar{c}$, respectively.

Our standpoint is to investigate the difference $\bar{V}_r - V_r$. The result is as follows:

(1.4)
$$\bar{V}_r^i - V_r^i = [W_{hj}^i(x,\xi,\eta)(\xi^j + \eta^j) + L_{hj}^i(x,\xi,\eta)(\eta^k - \xi^k) + K_{hjk}^i(x,\xi,\eta)\eta^j\xi^k]V^h + \cdots .$$

Case II. Let four points p, q, r, s be the same in Case I. However, curves c_3, c_4 are different from (I) as follows

$$(II)\begin{cases} c_1(t): & x^i(t) = x^i + t\xi^i \ (p \text{ to } q), \\ c_2(t): & x^i(t) = x^i + \xi^i + t\eta^i \ (q \text{ to } r), \\ c_3(t): & x^i(t) = x^i + \xi^i + \eta^i - t\xi^i \ (r \text{ to } s), \\ c_4(t): & x^i(t) = x^i + \eta - t\eta^i \ (s \text{ to } p), \end{cases}$$

where $0 \le t \le 1$.

We take a loop $c = c_1 + c_2 + c_3 + c_4(p \to q \to r \to s \to p)$ and consider a linear parallel displacement along c. Let $V = (V^i)$ be an initial vector at p and let V_q, V_r, V_s be the values of the parallel vector field along c at q, r, s, respectively. Further, let $\bar{V}$ be the value at the end point p.

Our standpoint is to investigate the difference $\bar{V} - V$. The result is as follows:

(1.5)
$$\begin{aligned} &\bar{V}^i - V^i \\ &= [W_{hj}^i(x,-\eta,\xi)(\xi^j + \eta^j) + L_{hj}^i(x,-\eta,\xi)(\eta^j - \xi^j) + K_{hjk}^i(x,-\eta,\xi)\xi^j\eta^k]V^h + \cdots . \end{aligned}$$

Remark 1.1 *In* (1.4) *and* (1.5), $(\cdots)$ *expresses 3rd and more order terms with respect to* ξ, η.

After all, we have the following theorem([9],[10]).

Theorem 1.1 *Let M be an n-dimensional differentiable manifold with a Finsler connection $F\Gamma = (N_j^i(x,y), F_{jk}^i(x,y), C_{jk}^i(x,y))$ satisfying $T_{jk}^i(x,y) = 0$, $D_j^i(x,y) = 0$. First, for an infinitesimal parallelogram defined by (I) and an initial vector $V = (V^i)$, we have the difference $\bar{V}_r - V_r$ satisfying* (1.4). *Next, for an infinitesimal parallelogram defined by (II) and an initial vector $V = (V^i)$, the parallel vector $\bar{V} = (\bar{V}^i)$ is obtained and the differences $\bar{V} - V$ satisfies* (1.5).

After that the author studied in detail properties of W, L, K in [10] and obtained the following propositions and theorem.

Proposition 1.1 *(Proposition 3.1 in [10]) Let $F\Gamma = (N_j^i(x,y), f_{hj}^i(x,y), C_{hj}^i(x,y))$ be a Finsler connection satisfying $T_{rj}^i = 0$ and $D_j^i = 0$. Then $W = 0$ on $\mathcal{H}$ is equivalent to $L = 0$ on $\mathcal{H}$.*

Proof

From $W^i_{hj}(x,y,z)=0$, $F^i_{hj}(x,y)=F^i_{hj}(x,z)$ is satisfied. This implies

$$\frac{\partial F^i_{hj}}{\partial y^k}(x,y)=\frac{\partial F^i_{hj}}{\partial z^k}(x,z)=0. \tag{1.6}$$

Therefore $L^i_{hj}(x,y,z)=0$ is satisfied.
Inversely, we assume $L^i_{hj}(x,y,z)=0$. Then the following equation

$$\frac{\partial F^i_{hm}}{\partial y^j}(x,y)z^m=-\frac{\partial F^i_{hm}}{\partial z^j}(x,z)y^m \tag{1.7}$$

is satisfied on any points (x,y,z). We take partial derivations by y^l and z^k of both sides, respectively. Then we have

$$\frac{\partial^2 F^i_{hk}}{\partial y^j\partial y^l}(x,y)=-\frac{\partial^2 F^i_{hl}}{\partial z^j\partial z^k}(x,z). \tag{1.8}$$

This means that the derivative of the second order by the second variable of the coefficient F^i_{hj} has no the second variable. Namely,

$$\frac{\partial F^i_{hj}}{\partial y^k}(x,y)=Q^i_{hjkm}(x)y^m \tag{1.9}$$

is satisfied.

On the other hand, $F^i_{hj}(x,y)$ is positively homogeneous of degree 0 with respect to the variable y. So $\frac{\partial F^i_{hj}}{\partial y^k}(x,y)y^k=0$ is satisfied. Therefore we have

$$Q^i_{hjkm}(x)y^my^k=0. \tag{1.10}$$

The above quadratic form of y is satisfied on any y so $Q^i_{hjkm}(x)=0$ must be true. Therefore we have $\frac{\partial F^i_{hj}}{\partial y^k}(x,y)=0$. Namely,

$$W^i_{hj}(x,y,z)=0 \tag{1.11}$$

is satisfied.

Q.E.D.

In addition, according to the above proof, we have the following proposition.

Proposition 1.2 *(Proposition 3.2 in [10]) Let $F\Gamma=(N^i_j(x,y),f^i_{hj}(x,y),C^i_{hj}(x,y))$ be a Finsler connection satisfying $T^i_{rj}=0$ and $D^i_j=0$. If $W=0$ is satisfied on $\mathcal{H}$, then $\frac{\partial F^i_{hj}}{\partial y^k}(x,y)=0$, namely, $F^i_{hj}=F^i_{hk}(x)$ is satisfied on TM.*

Further, if we assume $W^i_{hj}(x,y,z)=0$ and $K^i_{hjk}(x,y,z)=0$ on $\mathcal{H}$, then we can prove the following proposition.

Proposition 1.3 *(Proposition 3.3 in [10]) Let* $F\Gamma = (N^i_j(x,y), f^i_{hj}(x,y), C^i_{hj}(x,y))$ *be a Finsler connection satisfying* $T^i_{rj} = 0$ *and* $D^i_j = 0$. *For any point* (x,y,z) *on* $\mathcal{H}$, *if* $W^i_{hj}(x,y,z) = 0$ *and* $K^i_{hjk}(x,y,z) = 0$ *are satisfied, then the torsion tensor fields* $P^i_{hj}(x,y), R^i_{hj}(x,y), C^i_{hj}(x,y)$ *and the curvature tensor fields* $R^i_{hjk}(x,y), P^i_{hjk}(x,y)$ *of* $F\Gamma$ *satisfy the following equations:*

$$P^i_{hj}(x,y) = 0,\ R^i_{hj}(x,y) = 0,\ P^i_{hjk}(x,y) + C^i_{hk|j}(x,y) = 0,\ R^i_{hjk}(x,y) = 0. \tag{1.12}$$

Proof
Since Proposition 1.2, $\frac{\partial F^i_{hj}}{\partial y^k}(x,y) = 0$ is satisfied. From $P^i_{hj} = \frac{\partial N^i_h}{\partial y^j} - F^i_{jh}$ and $N^i_j = y^m F^i_{mj}(D=0)$,

$$P^i_{hj}(x,y) = 0 \tag{1.13}$$

is satisfied. Next, from $\frac{\partial F^i_{hj}}{\partial y^k} = P^i_{hjk} + C^i_{hk|j} - C^i_{hq}P^q_{jk}$ and $P^i_{hj} = 0$, we have

$$P^i_{hjk}(x,y) + C^i_{hk|j}(x,y) = 0. \tag{1.14}$$

And from $K^i_{hjk}(x,y,z) = 0$, of course $K^i_{hjk}(x,y,y) = 0$ is satisfied. In addition, in general, $K^i_{hjk}(x,y,y) = R^i_{hjk}(x,y) - C^i_{hm}(x,y)R^m_{jk}(x,y)$ is true. So $R^i_{hjk} - C^i_{hm}R^m_{jk} = 0$ is satisfied. And from $R^i_{jk} = y^m(R^i_{mjk} - C^i_{ms}R^s_{jk})$, we have

$$R^i_{jk}(x,y) = 0. \tag{1.15}$$

We apply the above conclusion in $R^i_{hjk} - C^i_{hm}R^m_{jk} = 0$ again, then we obtain

$$R^i_{hjk}(x,y) = 0. \tag{1.16}$$

Q.E.D.

For a Finsler space, the author stated in detail the conditions to be locally Minkowski space in [8]. If we apply Proposition 1.3 to the Finsler space with the property of h-metrical, then we have the following theorem.

Theorem 1.2 *(Theorem 3.2 in [10]) Let* $(M, F, F\Gamma)$ *be a Finsler space with the Finsler connection* $F\Gamma = (N^i_j(x,y), F^i_{jk}(x,y), C^i_{jk}(x,y))$ *satisfying* $T^i_{jk} = 0,\ D^i_j = 0$ *and* $g_{ij|k} = 0$.
If W *and* K *vanish on* $\mathcal{H}$, *then the Finsler space* $(M, F, F\Gamma)$ *is a locally Minkowski space and the inverse property is also true.*

2 The first and the second variations of arc lengths

Now we investigate the first and the second variations of arc lengths under the standpoint of linear parallel displacements. Let p, q be points on M and let $c(u)$ (u: arc length, $0 \leq u \leq L$) be a curve from p to q. I is an open interval including the closed interval $[0, L]$, where $p = c(0),\ q = c(L)$. Further $I_\epsilon = (-\epsilon, \epsilon)$ is an infinitesimal open interval. Then a variation $\alpha(u,v)$ is a differentiable map as follows:

$$\alpha : I \times I_\epsilon \longrightarrow M;\ ((u,v) \longrightarrow \alpha(u,v) = (x^i(u,v))), \tag{2.1}$$

where $\alpha(u,0) = c(u)$. Then $\alpha_v(u) := \alpha(u,v)$ with fixed v is called "variational curve". Further we denote vector fields $\frac{\partial\alpha}{\partial u} = \frac{\partial x^i}{\partial u}\frac{\partial}{\partial x^i}$ and $\frac{\partial\alpha}{\partial v} = \frac{\partial x^i}{\partial v}\frac{\partial}{\partial x^i}$ by ξ and η, respectively. Namely, we put $\xi = \frac{\partial\alpha}{\partial u}$ and $\eta = \frac{\partial\alpha}{\partial v}$. Especially, η is called "variational vector". In addition, we denote the coefficients of ξ and η by $\xi^i := \frac{\partial x^i}{\partial u}$ and $\eta^i := \frac{\partial x^i}{\partial v}$, respectively.

First, from $\frac{\partial^2 x^i}{\partial u\partial v} = \frac{\partial^2 x^i}{\partial v\partial u}$ we have very important relation as follows:

$$\frac{\partial\xi^i}{\partial v} = \frac{\partial\eta^i}{\partial u}. \tag{2.2}$$

And on the curve c, we also have the following equations

$$\xi = \dot{c} = \frac{dc}{du}, \quad F(c,\dot{c}) = \sqrt{g_{ij}(x,\dot{c})\dot{c}^i\dot{c}^j} = 1. \tag{2.3}$$

In addition, we use following notations

(2.4)

$$<\xi,\xi>_\xi := \| \xi \|_\xi^2 = g_{ij}(x,\xi)\xi^i\xi^j,\ <\eta,\eta>_\xi := \| \eta \|_\xi^2 = g_{ij}(x,\xi)\eta^i\eta^j,\ <\xi,\eta>_\xi := g_{ij}(x,\xi)\xi^i\eta^j.$$

From (2.3) and (2.4), on c, we have

$$<\dot{c},\dot{c}>_{\dot{c}} = \| \dot{c} \|_{\dot{c}}^2 = 1. \tag{2.5}$$

We assume that variation $\alpha(u,v)$ keeps the endpoints p,q fixed. Then variational vector $\eta(0,v) = \eta(L,v) = o$ are satisfied(See Figure 1). Here we put $L(v)$ as the arc length of a variational curve α_v as follows:

$$L(v) = \int_0^L F(\alpha, \frac{\partial\alpha}{\partial u})du = \int_0^L \sqrt{g_{ij}(x,\xi)\xi^i\xi^j}du. \tag{2.6}$$

Here we put the following definition.

Definition 2.1 *The quantity $L'(0) = \frac{dL}{dv}|_{v=0}$ is called "first variation" of $L(v)$ with respect to the variation α and the curve c which satisfies $L'(0) = 0$ is called "extremal(or critical) curve".*

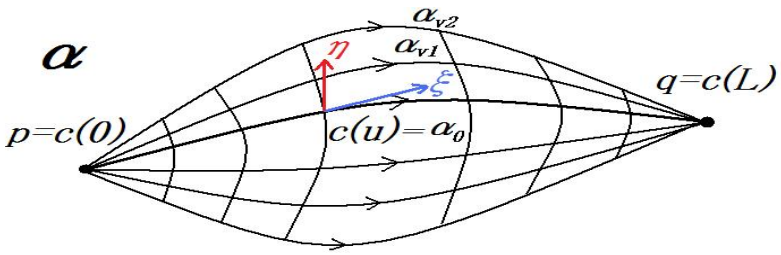

Figure 1: Variation

We consider the following variational problems:

1. Find extremal curve out!
2. Investigate the minimality of the arc length for the extremal curve!

For the above Problem 1, we need

$$L'(v) = \int_0^L F' du = \int_0^L \frac{\partial F}{\partial v} du. \tag{2.7}$$

If we use, for (2.7), the following formations (2.8) $\sim$ (2.13):

$$F' = \frac{\partial F}{\partial x^i}\frac{\partial x^i}{\partial v} + \frac{\partial F}{\partial \xi^i}\frac{\partial \xi^i}{\partial v} = \frac{\partial F}{\partial x^i}\eta^i + \frac{\partial F}{\partial \xi^i}\frac{\partial \eta^i}{\partial u}, \tag{2.8}$$

$$\frac{\partial g_{\lambda\mu}}{\partial y^i}(x,\xi)\xi^\lambda\xi^\mu = 0, \tag{2.9}$$

$$\frac{\partial F}{\partial x^i} = \frac{1}{2F}\frac{\partial F^2}{\partial x^i} = \frac{1}{2F}\frac{\partial}{\partial x^i}(g_{\lambda\mu}(x,\xi)\xi^\lambda\xi^\mu) = \frac{1}{2F}\frac{\partial g_{\lambda\mu}}{\partial x^i}(x,\xi)\xi^\lambda\xi^\mu, \tag{2.10}$$

$$\frac{\partial F}{\partial \xi^i} = \frac{1}{2F}\frac{\partial F^2}{\partial \xi^i} = \frac{1}{2F}\frac{\partial}{\partial \xi^i}(g_{\lambda\mu}(x,\xi)\xi^\lambda\xi^\mu) = \frac{1}{2F}(\frac{\partial g_{\lambda\mu}}{\partial y^i}(x,\xi)\xi^\lambda\xi^\mu + 2\xi_i) = \frac{1}{F}\xi_i, \tag{2.11}$$

$$\nabla_\xi \eta^i := \frac{\partial \eta^i}{\partial u} + F^i_{\alpha\beta}(x,\xi)\xi^\alpha\eta^\beta, \tag{2.12}$$

$$\xi_i = g_{ij}(x,\xi)\xi^j, \quad g_{ij|r} = 0. \tag{2.13}$$

then we can obtain

$$F' = \frac{1}{F}\xi_i \nabla_\xi \eta^i. \tag{2.14}$$

From $F' = \frac{1}{F}\xi_i\nabla_\xi\eta^i = \frac{1}{F}g_{ij}(x,\xi)\xi^j\nabla_\xi\eta^i = \frac{1}{F} < \xi, \nabla_\xi\eta >_\xi$, we obtain

$$L'(v) = \int_0^L \frac{1}{F} < \xi, \nabla_\xi \eta >_\xi du, \tag{2.15}$$

and, on curve c, from $v = 0$, $F = 1$ and $\xi = \dot{c}$, we have

$$L'(0) = \int_0^L < \dot{c}, \nabla_{\dot{c}}\eta >_{\dot{c}} du. \tag{2.16}$$

In addition, from $g_{ij|k} = 0$ and $\frac{\partial g_{ij}}{\partial y^k}(c,\dot{c})\dot{c}^i = 0$, we have $\frac{d}{du} < \dot{c}, \eta >_{\dot{c}} = < \nabla_{\dot{c}}\dot{c}, \eta >_{\dot{c}} + < \dot{c}, \nabla_{\dot{c}}\eta >_{\dot{c}}$. Therefore we can have

$$L'(0) = < \dot{c}, \eta >_{\dot{c}} |_0^L - \int_0^L < \nabla_{\dot{c}}\dot{c}, \eta >_{\dot{c}} du. \tag{2.17}$$

At the end point, $\eta(0,v) = \eta(L,v) = o$ are true. After all, we can obtain

$$L'(0) = -\int_0^L < \nabla_{\dot{c}}\dot{c}, \eta >_{\dot{c}} du. \tag{2.18}$$

Therefore the following proposition is true.

Theorem 2.1 *Geodesic*($\nabla_{\dot{c}}\dot{c} = 0$) *is an extremal curve. Inversely, for any variation with fixed endpoints, if the curve is extremal, then it is a geodesic.*

Next, let's investigate Problem 2. We must calculate the following expression.

$$L''(v) = \int_0^L F'' du. \tag{2.19}$$

If we use, for (2.19), the following formations (2.20) $\sim$ (2.23):

$$F' = \frac{1}{F} g_{ij}(x,\xi)\xi^j \nabla_\xi \eta^i,\ F'' = \frac{\partial}{\partial v}(\frac{1}{F} g_{ij}(x,\xi)\xi^j \nabla_\xi \eta^i), \tag{2.20}$$

$$\nabla_\eta \xi^\mu = \frac{\partial \xi^\mu}{\partial v} + F^\mu_{\alpha\beta}(x,\eta)\eta^\alpha \xi^\beta,\ \nabla_\xi \eta^\mu = \frac{\partial \eta^\mu}{\partial u} + F^\mu_{\alpha\beta}(x,\xi)\xi^\alpha \eta^\beta, \tag{2.21}$$

$$\frac{\partial \xi^\mu}{\partial v} = \frac{\partial \eta^\mu}{\partial u},\ \nabla_\xi \eta^\mu = \nabla_\eta \xi^\mu + W^\mu_{\alpha\beta}(x,\xi,\eta)\xi^\alpha \eta^\beta, \tag{2.22}$$

$$\nabla_\eta \nabla_\xi \eta^\lambda = \frac{\partial}{\partial v}(\nabla_\xi \eta^\lambda) + F^\lambda_{\alpha\beta}(x,\eta)\eta^\alpha \nabla_\xi \eta^\beta,\ \xi_\alpha = g_{\delta\alpha}(x,\xi)\xi^\delta,\ g_{ij|k} = 0, \tag{2.23}$$

then we can obtain

$$F'' = \frac{1}{F}\xi_\lambda \nabla_\eta \nabla_\xi \eta^\lambda + (\frac{1}{F} g_{\lambda\mu}(x,\xi) - \frac{1}{F^3}\xi_\lambda \xi_\mu)\nabla_\xi \eta^\mu \nabla_\xi \eta^\lambda + \Psi_{\beta\lambda}\eta^\beta \nabla_\xi \eta^\lambda, \tag{2.24}$$

where $\Psi_{\beta\lambda} = \frac{1}{F}(F^\alpha_{\beta\lambda}(x,\xi) - F^\alpha_{\beta\lambda}(x,\eta))\xi_\alpha = \frac{1}{F} W^\alpha_{\beta\lambda}(x,\xi,\eta)\xi_\alpha (\neq 0)$.

On the curve c, $v = 0$, $F = 1$ and $\xi = \dot{c}$ are true. So we have

$$F'' =< \dot{c}, \nabla_\eta \nabla_{\dot{c}} \eta >_{\dot{c}} + \parallel \nabla_{\dot{c}} Y \parallel^2_{\dot{c}} - < \dot{c}, \nabla_{\dot{c}} \eta >^2_{\dot{c}} + < \dot{c}, w >_{\dot{c}}, \tag{2.25}$$

where $w = (W^\alpha_{\beta\lambda}(c,\dot{c},\eta)\eta^\beta \nabla_{\dot{c}} \eta^\lambda)$.
Here, we need the following formation to arrange $< \dot{c}, \nabla_\eta \nabla_{\dot{c}} \eta >_{\dot{c}}$,

$$\nabla_\eta \nabla_{\dot{c}} \eta^k - \nabla_{\dot{c}} \nabla_\eta \eta^k.$$

According to linear parallel displacements along the infinitesimal parallelogram, we have the following formation

(2.26)
$$\nabla_\eta \nabla_{\dot{c}} \eta^k - \nabla_{\dot{c}} \nabla_\eta \eta^k = W^k_{hj}(c,\eta,\dot{c})(\eta^j + \dot{c}^j)\eta^h + L^k_{hj}(c,\eta,\dot{c})(\dot{c}^j - \eta^j)\eta^h + K^k_{hrj}(c,\eta,\dot{c})\dot{c}^r \eta^j \eta^h.$$

Let consider two parallel vector fields $\eta \longrightarrow \Pi_{\dot{c}}\eta \longrightarrow \Pi_\eta \Pi_{\dot{c}}\eta$ and $\eta \longrightarrow \Pi_\eta \eta \longrightarrow \Pi_{\dot{c}}\Pi_\eta \eta$ along an infinitesimal parallelogram. Then the difference $\Pi_\eta \Pi_{\dot{c}}\eta - \Pi_{\dot{c}}\Pi_\eta \eta$ is expressed as follows:

$$\begin{aligned} &\Pi_\eta \Pi_{\dot{c}}\eta - \Pi_{\dot{c}}\Pi_\eta \eta \\ &= [W^k_{hj}(c,\eta,\dot{c})(\eta^j + \dot{c}^j) + L^k_{hj}(c,\eta,\dot{c})(\dot{c}^j - \eta^j) + K^k_{hrj}(c,\eta,\dot{c})\dot{c}^r \eta^j]\eta^h + \cdots . \end{aligned}$$

Now we denote by r^k the difference $\nabla_\eta \nabla_{\dot{c}} \eta^k - \nabla_{\dot{c}} \nabla_\eta \eta^k$ in (2.26). Then, from $< \dot{c}, \nabla_\eta \nabla_{\dot{c}} \eta >_{\dot{c}} =< \dot{c}, \nabla_{\dot{c}} \nabla_\eta \eta >_{\dot{c}} + < \dot{c}, r >_{\dot{c}}$, (2.25) turns into the following formation

$$F'' =< \dot{c}, \nabla_{\dot{c}} \nabla_\eta \eta >_{\dot{c}} + \parallel \nabla_{\dot{c}} \eta \parallel^2_{\dot{c}} - < \dot{c}, \nabla_{\dot{c}} \eta >^2_{\dot{c}} + < \dot{c}, r + w >_{\dot{c}} . \tag{2.27}$$

Then we have

$$L''(0) = \int_0^L \left\{< \dot{c}, \nabla_{\dot{c}} \nabla_\eta \eta >_{\dot{c}} + \parallel \nabla_{\dot{c}} \eta \parallel^2_{\dot{c}} - < \dot{c}, \nabla_{\dot{c}} \eta >^2_{\dot{c}} + < \dot{c}, r + w >_{\dot{c}}\right\} du. \tag{2.28}$$

Here we put the following definition.

Definition 2.2 *The quantity $L''(0)$ in* (2.28) *is called "second variation" of $L(v)$ with respect to the variation α.*

Furthermore, from $\frac{d}{du} < \dot{c}, \nabla_\eta \eta >_{\dot{c}} = < \nabla_{\dot{c}} \dot{c}, \nabla_\eta \eta >_{\dot{c}} + < \dot{c}, \nabla_{\dot{c}} \nabla_\eta \eta >_{\dot{c}}$, we have
(2.29)

$$F'' = \frac{d}{du} < \dot{c}, \nabla_\eta \eta >_{\dot{c}} - < \nabla_{\dot{c}} \dot{c}, \nabla_\eta \eta >_{\dot{c}} + \| \nabla_{\dot{c}} \eta \|^2_{\dot{c}} - < \dot{c}, \nabla_{\dot{c}} \eta >^2_{\dot{c}} + < \dot{c}, r + w >_{\dot{c}} .$$

Therefore we can obtain
(2.30)

$$L''(0) = < \dot{c}, \nabla_\eta \eta >_{\dot{c}} |^L_0 - \int_0^L < \nabla_{\dot{c}} \dot{c}, \nabla_\eta \eta >_{\dot{c}} du + \int_0^L \left\{ \| \nabla_{\dot{c}} \eta \|^2_{\dot{c}} - < \dot{c}, \nabla_{\dot{c}} \eta >^2_{\dot{c}} + < \dot{c}, r + w >_{\dot{c}} \right\} du.$$

Here we add an assumption "c is geodesic", then we have

$$L''(0) = \int_0^L \left\{ \| \nabla_{\dot{c}} \eta \|^2_{\dot{c}} - < \dot{c}, \nabla_{\dot{c}} \eta >^2_{\dot{c}} + < \dot{c}, R(\dot{c}, \eta) >_{\dot{c}} \right\} du, \ R(\dot{c}, \eta) := r + w. \tag{2.31}$$

Furthermore, if $Z := \eta - < \dot{c}, \eta >_{\dot{c}} \dot{c}$, from $< \dot{c}, Z >_{\dot{c}} = 0$, then $\nabla_{\dot{c}} Z = \nabla_{\dot{c}} \eta - < \dot{c}, \nabla_{\dot{c}} \eta >_{\dot{c}} \dot{c}$ and $\| \nabla_{\dot{c}} Z \|^2_{\dot{c}} = \| \nabla_{\dot{c}} \eta \|^2_{\dot{c}} - < \dot{c}, \nabla_{\dot{c}} \eta >^2_{\dot{c}}$ are satisfied. After all we obtain

$$L''(0) = \int_0^L \left\{ \| \nabla_{\dot{c}} Z \|^2_{\dot{c}} + < \dot{c}, R(\dot{c}, \eta) >_{\dot{c}} \right\} du, \tag{2.32}$$

and

$$R^k(\dot{c}, \eta) = W^k_{hj}(c, \eta, \dot{c}) \eta^h (\eta^j + \dot{c}^j - \nabla_{\dot{c}} \eta^j) + L^k_{hj}(c, \eta, \dot{c}) \eta^h (\dot{c}^j - \eta^j) + K^k_{hrj}(c, \eta, \dot{c}) \eta^h \dot{c}^r \eta^j. \tag{2.33}$$

Then we have

Theorem 2.2 *Let $\alpha(u, v)$ be a variation that the curve c is a geodesic. Then the second variation $L''(0)$ satisfies the equation* (2.32) *with $R(\dot{c}, \eta)$ of* (2.33).

Theorem 2.3 *We assume* (2.32) *and* (2.33). *Then*
1. *If the space is Riemannian, then R consists of Riemannian curvature $K^k_{\alpha\beta i}(c)$ only, namely, $R^k = K^k_{\alpha\beta i}(c) \eta^\alpha \dot{c}^\beta \eta^i$.*
2. *If the space is locally Minkowski, then $R \equiv 0$ is satisfied.*
3. *The set of spaces which satisfy $R \equiv 0$ involves Riemannian spaces and locally Minkowski spaces.*

Remark 2.1 *If we use $\nabla_\eta \xi^\mu = \frac{\partial \xi^\mu}{\partial v} + F^\mu_{\alpha\beta}(x, \xi) \eta^\alpha \xi^\beta$ (usual manner), then R leads us to the well-known flag curvature.*

Next we put the following definition.

Definition 2.3 *We call the quantity $-\frac{< \dot{c}, R(\dot{c}, \eta) >_{\dot{c}}}{\| Z \|^2_{\dot{c}}}$ "sectional curvature" with respect to $\dot{c}$ and η at point c, and we denote it by $\rho(\dot{c}, \eta)_{\dot{c}}$. Namely,*

$$\rho(\dot{c}, \eta)_{\dot{c}} = - \frac{< \dot{c}, R(\dot{c}, \eta) >_{\dot{c}}}{\| Z \|^2_{\dot{c}}} \ (Z = \eta - < \dot{c}, \eta >_{\dot{c}} \dot{c}), \tag{2.34}$$

where if $Z = o$, then $\rho(\dot{c}, \eta)_{\dot{c}} = 0$. Then the second variation is modified to

$$L''(0) = \int_0^L \left\{ \| \nabla_{\dot{c}} Z \|_{\dot{c}}^2 - \| Z \|_{\dot{c}}^2 \rho(\dot{c}, \eta)_{\dot{c}} \right\} du. \tag{2.35}$$

In Riemannian geometry, there is the notion of "relatively minimal curve". So we put the notion in Finsler geometry as follows:

Definition 2.4 *For any variation of a curve c, which keeps endpoints fixed, if the arc length of c is always minimal for those of all variational curves, then c is called "relatively minimal curve" with respect to the endpoints.*

"We assume that variations have the property which its variational vector field on the curve c is linearly independent to the tangent vector field $\dot{c}$ of c at least one point."

Then we have

Theorem 2.4 *If $\rho(\dot{c}, \eta)_{\dot{c}} \leq 0$ is satisfied on M, then any geodesic c is relatively minimal curve with respect to any endpoints on it.*

Proof

We prove this property by a reduction to absurdity.
At first, from (2.35), we notice $L''(0) \geq 0$, $\| \nabla_{\dot{c}} Z \|_{\dot{c}}^2 \geq 0$ and $- \| Z \|_{\dot{c}}^2 \rho(\dot{c}, \eta)_{\dot{c}} \geq 0$.
If $L''(0) = 0$ is satisfied, then we have $\| \nabla_{\dot{c}} Z \|_{\dot{c}}^2 = 0$ and $- \| Z \|_{\dot{c}}^2 \rho(\dot{c}, \eta)_{\dot{c}} = 0$. From $\| \nabla_{\dot{c}} Z \|_{\dot{c}}^2 = 0$,

$$\nabla_{\dot{c}} Z = 0 \tag{2.36}$$

is satisfied. We notice that Z is a parallel vector field along c in the sense of linear parallel displacements from (2.36), and c is a geodesic. Therefore its norm $\| Z \|_{\dot{c}}$ is constant on c. At a start point, $\eta = 0$ is satisfied, so $\| Z \|_{\dot{c}} = 0$ holds good on c. Therefore Z is zero vector field on c. Then

$$\eta = < \dot{c}, \eta >_{\dot{c}} \dot{c} \tag{2.37}$$

is satisfied. Our assumption, however, can not permit (2.37) because Z and $\dot{c}$ are linearly independent at least one point on c.
Therefore

$$L''(0) > 0 \tag{2.38}$$

is satisfied.

This conclusion means that the geodesic c is minimal.

Q.E.D.

References

[1] S.-S. Chern and Z. Shen. *Riemann-Finsler geometry*, volume 6 of *Nankai Tracts in Mathematics*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.

[2] M. Matsumoto. *Foundations of Finsler geometry and special Finsler spaces*. Kaiseisha Press, Shigaken, 1986.

[3] M. Matsumoto. Finsler geometry in the 20th-century. In *Handbook of Finsler geometry. Vol. 1, 2*, pages 557–966. Kluwer Acad. Publ., Dordrecht, 2003.

[4] T. Nagano. Notes on the notion of the parallel displacement in Finsler geometry. *Tensor (N.S.)*, 70(3):302–310, 2008.

[5] T. Nagano. On the parallel displacement and parallel vector fields in Finsler geometry. *Acta Math. Acad. Paedagog. Nyházi.*, 26(2):349–358, 2010.

[6] T. Nagano. A note on linear parallel displacements in Finsler geometry. *Journal of the Faculty of Global Communication, University of Nagasaki*, 12:195–205, 2011.

[7] T. Nagano. On the conditions for Finsler spaces to be locally flat Riemannian spaces. *preprint*, 2011.

[8] T. Nagano. On the conditions to be locally Minkowski spaces. *Proceedings of the Conference RIGA 2011*, pages 209–216, 2011.

[9] T. Nagano. The quantities derived from linear parallel displacements along an infinitesimal parallelogram. *Journal of the Faculty of Global Communication, University of Nagasaki*, 13:129–141, 2012.

[10] T. Nagano. On the quantities W, L, K derived from linear parallel displacements in Finsler geometry. *Journal of the Faculty of Global Communication, University of Nagasaki*, 14:123–132, 2013.

Department of Information and Media Studies
University of Nagasaki
E-mail: `hnagano@sun.ac.jp`